〈論 文〉

Intergenerational Transfers in models with endogenous labor supply

by
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In this paper we employ a general equilibrium overlapping generations model with endogenous labor supply to explore how transfers between different generations affect the level of welfare of different generations. We consider the case in which both young and old age household are suppliers of labor.

We analyze three ways to finance the transfers: a lump-sum tax, a consumption tax or a labor income tax. We show that the golden rule applies in any case: the utility of all generations improves with transfers from the young to the old when the population growth rate surpasses the interest rate.

We also show that policies of pure transfers from the old to the young generation are not Pareto optimal for any of the analyzed type of taxes because the old generation living at the moment of the implementation of the transfer policy, always looses with the policy, at least when no other additional compensation is made to her.

Keywords: Intergenerational Transfers. Dynamic Efficiency. Overlapping Generations.

Introduction

One central topic in the economic research has the merits of the competitive equilibrium and the ways that governmental intervention is able to improve it. In recent years, the research has particularly focused on how the government commanding transfers between different generations that would not occur spontaneously in a free economy, can improve the welfare of all generations. The most commonly studied type of transfers is the one that is performed between two coexistent generations, from a young generation usually a worker to an old generation, a retiree. This is basically a pension system which has been called the pay-as-you-go system or also unfunded pension system.

In the pay-as-you-go pension system the government collects a tax from workers, and transfers it to the a retiree. For an individual, as long as, the received pension during the retirement compensates the loss due to the tax paid during the youth, the reform would be beneficial. The opportunity cost of the tax paid is given by the market interest amount. For that individual, if pensions surpass the tax in more than the interest amount, then the reform is beneficial. As long as the young generation's income expands at a higher rate than the interest rate, the government can collect the required tax and assure a higher rate of return than the prevailing in the market.

The implementation of a pay-as-you-go pension system makes the retiree living at the moment of the policy better off because they receive the pensions while they do not pay the tax. For the following generations, if the interest rate is lower than the rate of growth of income, the reform will be beneficial. As long as there is no terminating period all generations are benefited with the implementation of the pension system. Then it is said that the pay-as-you-go pension system can produce Pareto improving resource allocation. The so-called "golden rule" says then, that the competitive equilibrium economy can be improved with a policy of transfers from the young to the old generation when the population growth rate surpasses the market interest rate. Formal proofs under different frameworks appear in Azariadis (1995).

The discussion respect to the problems that arise with the implementation of this type of intergenerational transfers has been centered in the negative effect on private saving and capital formation. Another aspect is that declining birth rates tend to lower the rentability of the pay-as-you-go method. Problems of this kind have induced an analysis of the possibility of a transition from the pay-as-you-go system to a fully-funded system.

While Breyer (1989) showed in a model with lump-sum contributions and payments that the pay-as-you-go system of transfers is intergenerationally efficient, so that a change of the system would make at least one generation worse off, Homburg (1990) questioned this result, because in the real world contributions are not lump-sum. He employed a model with variable labor supply, income-dependent lump-sum pension payments to show that in this framework a Pareto-improving transition path from an unfunded to a funded system exists. The argument rests on the observation that contributions to the unfunded system are usually levied in the form of a tax on labor income and thus distort the labor-leisure decision making of the individuals, causing a deadweight loss. A change in the system would allow to remove this deadweight loss and the emerging surplus could be used to design a Pareto-improving transition. The authors proposal is to substitute the income tax by a lump-sum tax, which could then gradually be reduced to zero because of the gains in efficiency. Brunner (1996) argued that a Pareto-improving transition to a funded system is not possible because any instrument applied to the financing of pensions in the transition phase involves intragenerational redistribution. Then, when individuals differ, a Parteo-improvement condition requires that lump-sum contributions are in some way differentiated between individuals. However, this instrument is not available because the government does not have the precise knowledge necessary to identify individuals by their primary characteristic.

In our paper we attempt an extension of the intergenerational transfer issue in two directions. We try to pose the topic in a more general framework and consider also the possibility of a Pareto-improving system of transfers to the young generation. A transfer policy from the old to the young is discarded in the literature because the initial generation would be paying the tax and would not receive the subsidy. When analyzing the pension or transfers to the retiree problem this is quite obvious, but is doubtful when it is assumed that all generations work. It is not clear how the implementation of a system of transfers affects the wage rate determined in the labor market. For the old working generation, if in spite of paying the tax, labor income expands, the negative effect of the tax could be compensated. Then, one more objective of our paper is to clarify how the "golden rule" operates in models of endogenous labor supply in all periods.

In our analysis we employ, a general equilibrium overlapping generation model of two period life, without altruism and endogenous labor supply. Both the young and the old generation work in the production of a perishable consumption good.

We show that for policies of transfers from the young to the old generation, the famous "golden rule" also applies when both the young and the old work and this for any of the sources for financing the transfers, that is either with a lump sum tax, a consumption tax or a proportional labor income tax.

Our results could have been more interesting if they were in the opposite direction as obtained. We confirm that the golden rule cannot be applied in the contrary direction, that is, when the interest rate is higher than the population growth rate transfers to the young generation cannot produce a Pareto efficient policy. The basic reason is that, for the old generation living in the moment of the implementation of the transfers policy, even if the wage rate of the old generation increases in several of the cases we analyzed, it never compensates the reduction in income due to the tax.

The paper is organized as follows. First we present the model for the behavior of the enterprise and the behavior of the household. Then, we collect the basic results. We included most of the basic mathematical developments in the Appendices.

1.The Model

The demographic structure of the economy of our model is rather simple. There are a large amount of identical households that live two periods: the youth and the old age. The number of households grows at a constant rate $n>-1$. Each period two generations are alive, the "young generation" born in that period and the "old generation" born in the previous one.

We name "first generation" the generation born in period $t=1$, "second generation" the one born in period $t=2$, and so on. Besides, there is a "initial generation" born old and that coexists with the first generation. At that moment the initial generation is in her old age and the first generation in her youth. If we denote with H_0 the number of households of the "initial generation", the total number of households, N_0, that are alive at that moment are the addition of H_0 and $H_0(1+n)$, that is the number of households of the first generation:

The total number of households of the second period $t=2$ is:

In general, for any period the number of existent households is

Households supply labor both during the youth and the old age. Demand for labor comes from a large number of identical enterprises that produce a non-storable consumption good. In the production process, enterprises employ constant return to scale technology that combines the two types of labor. Finally, our model is a general equilibrium model in which markets operating under perfect competition are at equilibrium simultaneously.

One more important precision is that we consider a model of non-terminating period, that is the economy evolves continuously to the infinity.

In the following two sections we characterize the economic behavior of the enterprise and of the household.

1.1. The production of goods

The enterprise produces a non-storable consumption good, employing a technology that combines young workers and old workers. We specify a Cobb Douglas constant return to scale production function of the following type

$$A_t = \left(L_t^y\right)^{\delta_1}\left(L_{t-1}^o\right)^{\delta_2} \qquad (1)$$

with δ_1, $\delta_2>0$ and $\delta_1+\delta_2=1$

A_t is the volume of production during period t

L_t^y is the employment of young workers that belong to generation t

L_{t-1}^o is the employment of old workers that belong to the generation t-1.

We define the ratio of old to young units of work as ℓ_t and the volume of production in per units of young workers as a_t. Then, we have

$$\ell_t \equiv \frac{L^o_{t-1}}{L^y_t} \text{ and } a_t \equiv \frac{A_t}{L^y_t} \text{ or } a_t = (\ell_t)^{\delta_2} \tag{2}$$

The volume of labor being supplied in period t by the young workers will be denoted by N^y_t and by N^o_{t-1} for the old workers. As households are identical, N^y_t, is computed simply multiplying the total number of households N_t by the amount of units of labor supplied by each household ℓ^y_t.

$$N^y_t = \ell^y_t N_t \tag{3}$$

Similarly

$$N^o_{t-1} = \ell^o_{t-1} N_{t-1}. \tag{4}$$

Besides, total supply will equal total demand for each type of worker and, for example the ratio of old workers to young workers being employed in the economy can be computed as N^o_{t-1}/N^y_t, or

$$\frac{N^o_{t-1}}{N^y_t} = \frac{\ell^o_{t-1} N_{t-1}}{\ell^y_t N_t} = \frac{\ell^o_{t-1}}{(1+n)\ell^y_t} \tag{5}$$

Furthermore, as all enterprises are identical, the ratio of employment of old workers to young workers for each enterprise is equal and also equal to the ratio for the whole economy:

$$\ell_t = \frac{L^o_{t-1}}{L^y_t} = \frac{N^o_{t-1}}{N^y_t} = \frac{\ell^o_{t-1}}{(1+n)\ell^y_t} \tag{6}$$

According to the first order condition for profit maximization enterprises pay to workers a wage rate that equalizes the value of its marginal productivity, that is,

$$u^y_t = \frac{\partial A_t}{\partial \ell^y_t} = \delta_1 (L^y_t)^{\delta_1 - 1} (L^o_{t-1})^{\delta_2} \tag{7}$$

$$u^o_{t-1} = \frac{\partial A_t}{\partial \ell^o_t} = \delta_2 (L^y_t)^{\delta_1} (L^o_{t-1})^{\delta_2 - 1}$$

or, in per capita terms

$$u_t^y = \delta_1 \left(\frac{\ell_{t-1}^o}{\ell_t^y (1+n)} \right)^{\delta_2}$$

$$u_{t-1}^o = \delta_2 \left(\frac{\ell_t^y}{\ell_{t-1}^o} (1+n) \right)^{\delta_1} \tag{8}$$

A useful result in the mathematical developments is that

$$\frac{u_t^y}{u_{t-1}^o} = \frac{\delta_1}{\delta_2} \ell_t \tag{9}$$

It can be immediately checked that the value of production equals the total cost of production:

$$A_t = u_t^y L_t^y + u_{t-1}^o L_{t-1}^o \tag{10}$$

or in per units of worker

$$a_t = u_t^y + u_{t-1}^o \ell_t \tag{11}$$

1.2. The market of goods

Total supply of consumption goods A_t is demanded by the young A_t^y and the old generation A_{t-1}^o.

$$A_t = A_t^y + A_{t-1}^o \tag{12}$$

We can also express the equilibrium condition in the market of goods in per capita terms as follows

$$\frac{A_t}{N_t^y} \frac{N_t^y}{N_t} = \frac{A_t^y}{N_t} + \frac{A_{t-1}^o}{N_{t-1}} \frac{N_{t-1}}{N_t} \tag{13}$$

$$a_t \ell_t^y = a_t^y + \frac{a_{t-1}^o}{1+n} \tag{14}$$

1.3. The behavior of the household

Each household of generation maximizes the present value of the utility that enjoys from consumption of goods and leisure during the youth (a_t^y and ℓe_t^y respectively) and old age (a_t^o and ℓe_t^o). We assume that the household is non-altruistic, that is, that does not care at all neither about other households nor about other generations.

Each period, the household is endowed with a constant volume of time (normalized to the unity) which she can enjoy as free time or leisure or spend it working. Labor supply during the youth and the old age are ℓ_t^y and ℓ_t^o respectively. Then, the following two relations hold

$$\begin{aligned} \ell_t^y + \ell e_t^y &= 1 \\ \ell_t^o + \ell e_t^o &= 1 \end{aligned} \qquad (15)$$

Total income of the household comes from income from labor during the youth $u_t^y \ell_t^y$ and/or the old age $u_t^o \ell_t^o$. Outlays are the consumption of goods and the payment of taxes.

We consider three types of taxes: the lump sum tax (π is the tax paid by the young and - ε the one paid by the old), the consumption tax, (τ is the consumption tax rate) and the labor income tax ($\tau_y$is the labor income tax rate of young workers and τ_0 the labor income tax of the old workers).

Transfers to the old generation is denoted by ε and transfers to the young by - π . Then, the budget constraint of the household can be written as

$$u_t^y(1-\tau_y)\ell_t^y - \pi + \frac{u_t^o(1-\tau_o)\ell_t^o + \varepsilon}{R_{t+1}} = a_t^y(1+\tau) + \frac{a_t^o(1+\tau)}{R_{t+1}} \qquad (16)$$

R_{t+1}, the interest factor, is defined as the unity plus the interest rate.

We specify the present value of the utility enjoyed during the youth and old age as separable functions

$$U_t(a_t^y, \ell e_t^y, a_t^o, \ell e_t^o) = U_t^y(a_t^y, \ell e_t^y) + U_t^o(a_t^o, \ell e_t^o) \quad (17)$$

using CES functions of the following type:

$$U_t^y = \frac{\alpha_y (a_t^y)^{1-\sigma} + \beta_y (\ell e_t^y)^{1-\sigma}}{1-\sigma} \quad (18)$$

$$U_t^o = \frac{\alpha_o (a_t^o)^{1-\sigma} + \beta_o (\ell e_t^o)^{1-\sigma}}{1-\sigma}$$

$\sigma > 0$ is a parameter($\sigma \neq 1$) which can be interpreted as the reciprocal of the elasticity of substitution between consumption of goods and leisure. This is because, the marginal rate of substitution is

$$MRS = -\frac{d(\ell e_t^y)}{d(a_t^y)} = \frac{\dfrac{\partial U_t^y}{\partial a_t^y}}{\dfrac{\partial U_t^y}{\partial \ell e_t^y}} = \frac{\alpha_y}{\beta_y}\left(\frac{\ell e_t^y}{a_t^y}\right)^{\sigma} \quad (19)$$

and the elasticity of substitution, defined as $ES = \dfrac{\dfrac{d\left(\dfrac{\ell e_t^y}{a_t^y}\right)}{\ell e_t^y / a_t^y}}{\dfrac{dMRS}{MRS}} = \dfrac{1}{\sigma}$ becomes $ES = \dfrac{1}{\sigma}$.

When $\sigma = 1$, the utility function degenerates into a Cobb-Douglas type function. We will present separately the results for this case.

On the other hand, α_y, $\alpha_o > 0$ and β_y, β_o are also parameters of the utility function.

We first present the effects of intergenerational transfers for the simple case in the household works all her endowment of time both during the youth and during the old age. This will mean that $\ell e_t^y = \ell e_t^o = 0$ and $\beta_y = \beta_o = 0$. Second, the case in which the household works all her time during the youth but consumes a positive amount of leisure during the old age. This will mean that $\ell e_t^y = 0$, $\ell e_t^o > 0$ and $\beta_y = 0$ and $\beta_o > 0$. Finally, we include some results for the case in which $\beta_y>0$ and $\beta_o>0$.

The household chooses the level of consumption of goods and leisure for her whole life span a_t^y, a_t^o, ℓe_t^y and ℓe_t^o, trying to maximize her utility, under the restriction of her budget and the restriction of the endowment of time. If we introduced the restriction given by the endowment of time in the budget constraint, we can formally present the optimization problem of the households as follows:

$$W_t = \underset{(a_t^y, a_t^o, \ell e_t^y, \ell e_t^o)}{Max} \quad U_t = \frac{\alpha_y (a_t^y)^{1-\sigma} + \beta_y (\ell e_t^y)^{1-\sigma} + \alpha_o (a_t^o)^{1-\sigma} + \beta_o (\ell e_t^o)^{1-\sigma}}{1-\sigma} \tag{20}$$

$$s.t. \quad u_t^y(1-\tau_y) - \pi + \frac{u_t^o(1-\tau_o)+\varepsilon}{R_{t+1}} = a_t^y(1+\tau) + u_t^y(1-\tau_y)\ell e_t^y + \frac{a_t^o(1+\tau) + u_t^o(1-\tau_o)\ell e_t^o}{R_{t+1}}$$

In order to find the solution of the above optimization problem, we can first solve the budget constraint for, for example, a_t^y, and substitute the obtainable expression into the utility function. Then, find the first order condition for maximization, computing the derivatives of the utility with respect to the other three variables

$$\frac{\partial U_t}{\partial \ell e_t^y} = 0 \quad , \quad \frac{\partial U_t}{\partial a_t^o} = 0 \quad and \quad \frac{\partial U_t}{\partial \ell e_t^o} = 0 \tag{21}$$

Saving during the youth, s_t, is defined as the difference between after tax income and the value of consumption:

$$s_t = u_t^y(1-\tau_y) - \pi - a_t^y(1+\tau) - u_t^y(1-\tau_y)\ell e_t^y \tag{22}$$

This amount of saving is supplied in the capital market, the household would receive an interest and the produce will be additional income in the old age. The household is non-altruistic, then, during the old age, before dying she consumes all her disposable income:

$$s_t R_t + u_t^o(1-\tau_o)\ell_t^o + \varepsilon = a_t^o(1+\tau) \qquad (23)$$

Using equation (21) and the budget constraint we can solve for the four choice variables of the household.

1.4. The saving behavior of the economy

In our model the household lives only two periods, then, the only economic agents that participate in the capital market are the young households. The old generation dies once the current period finishes and as the household is non-altruistic she does not leave any bequest to the following generation then she consumes all her endowment.

Then, saving of the economy is formed from the aggregation of the saving of the young households exclusively. Equilibrium in the capital market imply that the aggregated net saving is zero and with identical households saving of each household is zero

$$u_t^y(1-\tau_y)\ell_t^y - \pi - a_t^y(1+\tau) = 0 \qquad (24)$$

The budget of the household in the old age will become as follows:

$$u_t^o(1-\tau_o)\ell_t^o + \varepsilon = a_t^o(1+\tau) \qquad (25)$$

Then, each household consumes her own endowment and autarky prevails.

1.5. The complete model

A complete and compact version of the model is presented below. The first three equations correspond to the production function and the first order conditions for profit maximization of enterprises. The fourth, fifth and sixth equations are the first order conditions for optimization of the household. Then, the following three equations are the budget constraint and time endowment constraint of the households. Equation ten and eleven are the equilibrium condition in the market of goods and in the market of capital. The budget constraint of the government is the following equation. The last equation is simple the definition of the variable proportion of both types of labor.

$$a_t = a_t\left(\ell_t^y, \ell_{t-1}^o\right) \tag{m-1}$$

$$u_t^y = u_t^y\left(\ell_t^y, \ell_{t-1}^o\right) \tag{m-2}$$

$$u_{t-1}^o = u_{t-1}^o\left(\ell_t^y, \ell_{t-1}^o\right) \tag{m-3}$$

$$\frac{\partial U_t}{\partial \ell e_t^y} = 0 \tag{m-4}$$

$$\frac{\partial U_t}{\partial a_t^o} = 0 \tag{m-5}$$

$$\frac{\partial U_t}{\partial \ell e_t^o} = 0 \tag{m-6}$$

$$u_t^y\left(1-\tau_y\right)\ell_t^y - \pi + \frac{u_t^o\left(1-\tau_o\right)\ell_t^o + \varepsilon}{R_{t+1}} = a_t^y\left(1+\tau\right) + \frac{a_t^o\left(1+\tau\right)}{R_{t+1}} \tag{m-7}$$

$$\ell_t^y + \ell e_t^y = 1 \tag{m-8}$$

$$\ell_t^o + \ell e_t^o = 1 \tag{m-9}$$

$$a_t \frac{1+n}{2+n} = a_t^y + \frac{a_{t-1}^o}{1+n} \tag{m-10}$$

$$u_t^y\left(1-\tau_y\right)\ell_t^y - \pi - a_t^y\left(1+\tau\right) = 0 \tag{m-11}$$

$$\ell_t = \frac{\ell_{t-1}^o}{(1+n)\ell_t^y} \tag{m-12}$$

$$u_t^y \ell_t^y \tau_y + u_{t-1}^o \frac{\ell_{t-1}^o}{1+n}\tau_o + \tau\left(a_t^y + \frac{a_{t-1}^o}{1+n}\right) + \pi = \frac{\varepsilon}{1+n} \tag{m13}$$

There are thirteen equations and thirteen endogenous variables:

$$a_t, \ell_t, \ell_{t-1}^o, \ell_t^y, u_{t-1}^o, u_t^y, \ell e_{t-1}^o, \ell e_t^o, \ell e_t^y, a_{t-1}^o, a_t^o, a_t^y, R_{t+1}$$

Notice that we do not have a separate equation indicating the household consumes all her disposable income in the old age, that is

$$u_t^o(1-\tau_o)\ell_t^o + \varepsilon = a_t^o(1+\tau) \qquad (26)$$

because it can be easily obtained from the other equations of the model.

2. Intergenerational transfers and efficiency

In this section we collect the basic results, the mathematical proof are included in the appendices.

The basic question is to evaluate whether a certain type of transfer from the young to the old, and vice-versa allows for a Pareto-improving resource allocation, that is allows the utility of all generations to improve.

First we present the case of exogenous labor supply during both the youth and the old age. Then, we analyze the case of endogenous labor supply only in the old age; finally, the case of endogenous labor supply in both periods.

2.1. Exogenous labor supply

In the case the supply of labor of the household is exogenous, the wage rates become fixed and, then, the model works similarly to a non-production economy (Azariadis, 1995). As the wage rates will not change with the transfer policy, transfers from the old to the young will always reduce the level of utility of the household of the initial generation if compare to the non-interventionist case.

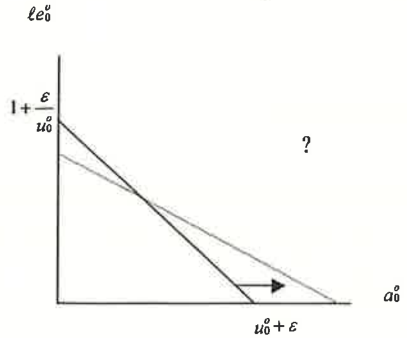
As shown in Appendix 1 for any of the three types of taxes, transfers from the young to the old generation improve the level of utility of all generations if under non intervention the population growth rate is higher than the market interest rate. Furthermore, the maximum level of utility of the household of all generations is achieved setting a tax rate that makes the interest rate equalize the population growth rate.

2.2. Endogenous labor supply in the old age

When we consider the labor supply endogenous in at least one of the two periods, the analysis of a policy of transfers from the old to the young becomes relevant and could not be discarded immediately as we could do in the case of exogenous labor supply. Let's see the intuition that says that there could be some possibility for the policy to be Pareto efficient. Specifically, we would like to know whether the initial generation is always hurt. For this, let us use the case of lump sum transfers.

$$\underset{\left(a_0^o, \ell e_0^o\right)}{Max} \quad U_0^o = U_0^o\left(a_0^o, \ell e_0^o\right)$$
$$s.t. \qquad u_0^o + \varepsilon = a_0^o + u_0^o \ell e_0^o$$

If due to the transfers the budget set expands at least in some relevant portion, then utility could improve in certain circumstances. As $\varepsilon = \pi(1+n)$, the budget line can also be written as $1+\frac{\pi(1+n)}{u_0^o}-a_0^o\frac{1}{u_0^o}=\ell e_0^o$. Notice that if, for example, we compare with a situation with zero taxes, the implementation of a policy of transfers to the young (ε <0) will imply that the intercept for the variable leisure will reduce. On the other hand, the intercept for the consumption of goods though would become lower due to the direct expansion of the tax. Whether the budget line moves to the right in some area or on the contrary, shifts inwards, will depend on what happens with the wage rate. If due to the policy the wage rate expands enough, the budget line could rotate and the set of available choices that the household faces could expand in a certain area.



As we show in Appendix 1, it is not obvious neither how the wage rate reacts nor how big this reaction is and then, whether the negative effect of the tax can be compensated or not. This justifies a general equilibrium approach that allows to observe the effect of the transfers policy on wage rates.

We collect here the main propositions developed in the Appendices and summarize them in the following two

Proposition I: The utility of the initial generation increases with transfers from the young to the old generation for any of the three tax instruments: a lump-sum tax a consumption tax and a proportional labor income tax.

Corollary: A policy of transfers to the young is not Pareto efficient.

Proposition II: The utility of the generations after the initial increases with transfers from the young to the old if and only if the interest rate is lower than the population growth rate. And this is valid for any of the three types of taxes.

2.3. Endogenous labor supply in the youth and in the old age

We summarize the main proposition next

Proposition III: Transfers to the young generation does not produce a Pareto optimal policy because the utility or the initial generation falls at least in the case of a consumption tax or a labor income tax.

In the case of the lump sum tax we could not obtain a definite answer. We left this case for further future analysis.

3. Conclusions and directions for future work

In this paper we attempted a wider approach to the analysis of the problem of the intertemporal efficiency of the market economy. We show that in the framework of an overlapping generations model with endogenous labor supply utility of all generations improves with a policy of transfers from the young to the old even when both young and old age household work. On the contrary a policy of transfers from the old to the young cannot produce a Pareto-improving because the old generation always looses with the reform. These results are all valid for any of the three ways of financing the transfers: a lump-sum tax, a consumption tax or a labor income tax.

Two basic suggestions for future work. First of all, it would be of interest to evaluate how the inclusion of capital as a factor of production could affect the results. The second interesting line of work is how parallel compensating measures could avoid the initial generation from loosing in the case of transfers to the young generation and then allow for a Pareto-improving transfer policy.

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Mathematical Appendices1

APPENDIX 1

Intergenerational transfers in a model with exogenous labor supply

In this case, we assume labor supply is exogenous both in the in the youth and in the old age. As leisure time is also exogenously given, $\beta_y = \beta_o = 0$. We normalize the labor supply to the unity in both periods. The optimization problem of the household presented in the main text will simplify as follows

$$W_t = \underset{(a_t^y, a_t^o)}{Max} \quad U_t = \frac{\alpha_y (a_t^y)^{1-\sigma} + \alpha_o (a_t^o)^{1-\sigma}}{1-\sigma}$$

$$s.t. \quad u_t^y(1-\tau_y) - \pi + \frac{u_t^o(1-\tau_o)+\varepsilon}{R_{t+1}} = a_t^y(1+\tau) + \frac{a_t^o(1+\tau)}{R_{t+1}}$$

When labor supply is exogenous for both types of workers, the wage rates become fixed and the model resembles the model of an exchange economy in which the household is assumed to be endowed with a certain amount of consumption goods in both periods of her life (Azariadis, 1995).

In order to find a solution to the maximization problem we can solve the budget constraint for a_t^y and substitute it in the first summation term of the utility function. Then, the problem transforms into a free maximization problem in which the variable of choice is a_t^o. The first order condition for maximization give us

$$a_t^o = a_t^y \left(\frac{\alpha_o}{\alpha_y} R_{t+1} \right)^{\frac{1}{\sigma}}$$

Using the above equation in the budget constraint we can find the demand for goods during the youth and during the old age.

Saving during the youth, s_t, is defined as the difference between after tax income and the value of consumption:

$$s_t = u_t^y(1-\tau_y) - \pi - a_t^y(1+\tau)$$

This amount of saving is supplied in the capital market. As only the young households participate in the capital market and as all households are identical, equilibrium in the capital market can be represented by

$$u_t^y(1-\tau_y) - \pi - a_t^y(1+\tau) = 0$$

It is immediate from the above that there will actually be no transactions in the capital market. Each household will consume her own endowment and autarky will prevail. The implicit market interest factor can be computed from the equilibrium equation of the capital market:

$$R = \frac{\alpha_y}{\alpha_o}\left(\frac{\delta_2(1+n)^{\delta_1}(1-\tau_o)+\varepsilon}{(1-\tau_y)\delta_1\left(\frac{1}{1+n}\right)^{\delta_2}-\pi}\right)^{\sigma}$$

As the interest factor becomes a function of parameters and taxes only, we obviate the time subscript.

The indirect utility function of the household

Before computing the effects of the different transfer policies let's compute the indirect utility function of the household.

$$W_t = \frac{\alpha_y\left((1-\tau_y)\delta_1\left(\frac{1}{1+n}\right)^{\delta_2}-\pi\right)^{1-\sigma} + \alpha_o\left(\delta_2(1+n)^{\delta_1}(1-\tau_o)+\varepsilon\right)^{1-\sigma}}{1-\sigma}(1+\tau)^{\sigma-1}$$

The above expression is valid for any generation from the first. The indirect utility function of the initial generation is

$$W_0 = \frac{\alpha_o\left(\delta_2(1+n)^{\delta_1}(1-\tau_o)+\varepsilon\right)^{1-\sigma}}{1-\sigma}(1+\tau)^{\sigma-1}$$

It can be directly observed from the above, that, for whatever policy instrument, the utility of the initial generation fall with an increase in any of the taxes. Therefore, Pareto efficient transfer policy should be based on transfers to the old generation. If not the initial generation looses.

1. Transfers to the old financed by a lump sum tax

When transfers to the old generation are being financed with a lump sum tax paid by the young generation, then $\tau_y = \tau_o = \tau = 0$ and the budget constraint of the government is $\varepsilon = (1+n)\pi$ with $\pi > 0$.

The effects of the policy on the utility of the household can be evaluated computing the derivative of the indirect utility function $W_t(\pi)$ with respect to the tax parameter. After some manipulations we get that the necessary and sufficient condition for the utility of the first and following generations to improve when transfers from the young to the old are financed with a lump sum tax. that is, when the population growth rate surpasses the interest rate:

$$\frac{dW_t}{d\pi} \geq 0 \Leftrightarrow R \leq 1+n$$

2. Transfers to the old financed by a consumption tax

When transfers to the old generation are being financed with a consumption tax. Then, $\tau_y = \tau_o = \pi = 0$ and the budget constraint of the government $\tau(a_t^y(1+n)+a_t^o-1)=\varepsilon$.

As $a_t^o = \dfrac{u_t^o+\varepsilon}{1+\tau}$, $a_t^y = \dfrac{u_t^y}{1+\tau}$ and $\dfrac{u_t^y}{u_{t-1}^o} = \dfrac{\delta_1}{\delta_2}\dfrac{1}{1+n}$ we get $\tau(1+n)^{\delta_1} = \varepsilon$

Computing $dW_t(\tau)/d\tau$ we get that the necessary and sufficient condition for utility improvement of the first and following generations is the following

$$\frac{dW_t}{d\tau} \geq 0 \Leftrightarrow 1+n > R$$

Then, the utility of all generations after the initial increases with transfers to the old generation financed with a consumption tax, when the population growth rate surpasses the interest rate.

3. Transfers using the labor income tax

When both the young and old workers spend one hundred percent of their endowment working, the wage rates become fixed and taxes on labor income are equivalent to lump sum taxes and we will have similar results as before. The optimal tax rates are fixed to make the interest rate equalize the population growth rate.

Appendix 2

Intergenerational transfers in a model with endogenous labor supply in the old age

In this case, we assume that the household supplies a constant volume of work during the youth but in the old age decides how much time to work maximizing her utility function. We normalize the exogenous supply of labor during the youth to the unity and make β_y=0. The optimization problem can be written as follows

$$\underset{(a_t^y, a_t^o, \ell e_t^o)}{Max} \quad U_t = \frac{\alpha_y (a_t^y)^{1-\sigma} + \alpha_o (a_t^o)^{1-\sigma} + \beta_o (\ell e_t^o)^{1-\sigma}}{1-\sigma}$$

$$s.t. \qquad u_t^y(1-\tau_y) - \pi + \frac{u_t^o(1-\tau_o)+\varepsilon}{R_{t+1}} = a_t^y(1+\tau) + \frac{a_t^o(1+\tau)+u_t^o(1-\tau_o)\ell e_t^o}{R_{t+1}}$$

One way to solve the above optimization problem is to solve the budget constraint for a_t^y, substitute it into the objective function and then compute the first order conditions

for utility maximization, $\dfrac{\partial U_t}{\partial a_t^o} = 0$ and $\dfrac{\partial U_t}{\partial \ell e_t^o} = 0$.

These two equations will give the following relation: $\dfrac{\beta_o}{\alpha_o}\dfrac{1}{u_t^o}\dfrac{(1+\tau)}{(1-\tau_o)} = \left(\dfrac{\ell e_t^o}{a_t^o}\right)^{\sigma}$.

Besides, when old, the household consumes all her endowment, then:

$$u_t^o(1-\tau_o)(1-\ell e_t^o) + \varepsilon = a_t^o(1+\tau)$$

Using these last two equations and the definition of the wage rate paid to old workers $u_t^o = \delta_2\left(\dfrac{1+n}{\ell_t^o}\right)^{\delta_1}$;we get equation $S(\ell e_t^o)$=0, that implicitly gives the solution for ℓe_t^o:

Then, the demand for leisure in the old age is given by

$$S\left(\ell e_t^o\right) \equiv \frac{\beta_o}{\alpha_o} \frac{(1+\tau)^{1-\sigma}}{(1-\tau_o) u_t^o} - \left(\frac{\ell e_t^o}{u_t^o(1-\tau_o)(1-\ell e_t^o)+\varepsilon}\right)^{\sigma}$$

and the supply of labor is $\ell_t^o = 1 - \ell e_t^o$.

Once the solution for the leisure of the old is found from the previous equation, we could find the solution for a_t^o using

$$a_t^o = \ell e_t^o \left(\frac{\alpha_o}{\beta_o} u_t^o \frac{(1-\tau_o)}{(1+\tau)}\right)^{1/\sigma}$$

Then, the solution for can be found using the equilibrium condition in the capital market: $u_t^y(1-\tau_y)-\pi = a_t^y(1+\tau)$. Notice that here, $u_t^y = \delta_1\left(\frac{\ell_{t-1}^o}{1+n}\right)^{\delta_2}$,and , $\ell_{t-1}^o = 1-\ell e_{t-1}^o$, are already determined.

Next, we find an expression for the indirect utility function that will be useful afterwards when trying to determine the effects of transfers.

Using the first order conditions: $a_t^o = \left[\frac{\alpha_o}{\beta_o} u_t^o \frac{(1-\tau_o)}{(1+\tau)}\right]^{\frac{1}{\sigma}} \ell e_t^o$ and $a_t^y = \frac{u_t^y(1-\tau_y)-\pi}{(1+\tau)}$

the indirect utility function of the household can be written as

$$W_t = \frac{\alpha_y\left(\frac{u_t^y(1-\tau_y)-\pi}{(1+\tau)}\right)^{1-\sigma} + \alpha_o\left(\frac{\alpha_o}{\beta_o} u_t^o \frac{(1-\tau_o)}{(1+\tau)}\right)^{\frac{1-\sigma}{\sigma}} \left(\ell e_t^o\right)^{1-\sigma} + \beta_o\left(\ell e_t^o\right)^{1-\sigma}}{1-\sigma}$$

In the above we take ℓe_t^o, u_t^y and u_t^o as the solution of the model, and thus being function of parameters and taxes.

In the case of exogenous labor supply in both periods, observing the indirect utility function of the household is enough to conclude that the only possible Pareto optimal transfer policy is from the young to the old. In the case labor supply is endogenous in at least one period, the effects of intergenerational transfers are less obvious because the wage rates will change with the policy. In the following sections we analyze the effects of transfers between generations in both directions, that is form the young to the old and from the old to the young.

1. Transfers financed by a lump sum tax

When transfers to the old generation are being financed by a lump sum tax, $\tau_y = \tau_o = \tau = 0$ and then, the budget of the government becomes

$$\pi(1+n) = \varepsilon$$

When $\pi > 0$, the policy implies a transfer from the young to the old. When $\pi < 0$, the policy implies a transfer from the old to the young.

We would like to compute the effects of those transfers on the level of consumption of goods and the leisure time, and more precisely on the final level of utility. It could be illustrative to consider what would happen in a partial equilibrium analysis, considering only the effects of the effects of the transfer policies on the behavior of an individual household. We could analyze this considering the demand functions for leisure and goods and computing $\frac{\partial \ell e_t^o}{\partial \pi}$, $\frac{\partial a_t^o}{\partial \pi}$ and $\frac{\partial a_t^y}{\partial \pi}$. For example, is immediate that. $\frac{\partial \ell e_t^o}{\partial \pi} > 0$ and $\frac{\partial a_t^o}{\partial \pi} > 0$, and $\frac{\partial a_t^y}{\partial \pi} < 0$.

Nevertheless, these are only the direct effects of the transfer policy because all prices of the economy, in particular, the wage rates, will also change. For example, as due to the direct effect, old workers supply of all households reduces and so will reduce the total supply in the labor market, then, the wage rate of old workers would increase in the market. How will react the household before any change in the wage rates provoked by the new policy?

$$\frac{\partial \ell e_t^o}{\partial u_t^o} = \frac{\left(\frac{\alpha_o}{\beta_o} u_t^o\right)^{\frac{1}{\sigma}} \left[1 - \frac{1}{\sigma} \frac{1}{u_t^o}\left(u_t^o + \varepsilon\right)\right] - \varepsilon}{\left[\left(\frac{\alpha_o}{\beta_o} u_t^o\right)^{\frac{1}{\sigma}} + u_t^o\right]^2}$$

When $\sigma < 1$, $\frac{\partial \ell e_t^o}{\partial u_t^o} < 0$, and for $\sigma > 1$ $\frac{\partial \ell e_t^o}{\partial u_t^o}$ could be positive or negative,

but will be positive, at least, for very small values of the tax parameter. This also says that, when the wage rate increases, the labor supply increases when σ <1 and fall when σ >1.

Then, while the direct effect of the change in the tax on the demand of leisure is

positive, $\frac{\partial \ell e_t^o}{\partial \pi} > 0$, the indirect effect through any resultant change in the wage rate

could be positive or negative. Whether the wager rate increases or decreases depends on how labor demand and labor supply interacts in the market.

For example in the market for old workers, the demand is a decreasing function of the wage rate while supply will an increasing or decreasing function depending on the value of σ . When $\sigma < 1$ $\frac{\partial \ell_t^o}{\partial u_t^o} > 0$, then transfer policy implies an increase in the wage rate. When, σ >1, the transfer policy will imply an increase or a reduction in the wage rate depending on which is higher, the slope of the demand or the supply function.

Is there any way of evaluating the effects of the policy without computing the equilibrium values of the variables? For example, is it possible to find out the final effects on employment, leisure and consumption simply observing the slopes of the supply and demand functions?

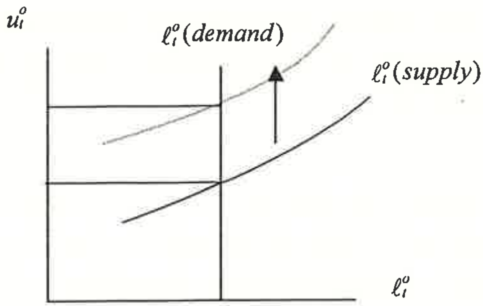
In particular, when analyzing the possibility of Pareto efficient policies of transfers from the old to the young, we would like to evaluate if, for the initial generation, the reduction in income due to the transfers could be surpassed by an increase in the wage rate. In order to give an answer to this question we must compute the general equilibrium and then observe the effects of the transfer policy.

Before this, we show that unless we compute the general equilibrium itself we cannot give a final answer.

Notice that, for example, in the labor market for old workers, the highest change in the wage rate due to the introduction of a transfer system, will be for an inelastic labor demand. We may take the case of inelastic labor demand as the upper bound for the change in the wage rate. Then, for this case, if the old generation makes transfers to the young the supply function of labor will shift upwards. This can be seen by simple inspection of the labor supply:

$$\ell_t^o = 1 - \frac{u_t^o + \varepsilon}{\left[\frac{\alpha_o}{\beta_o} u_t^o\right]^{\frac{1}{\sigma}} + u_t^o}$$

Remember that in the case of transfers to the young ε <0.



We show that even for the bound, for which the change in the wage rate is the highest we cannot give a final answer unless we compute the general equilibrium itself. Only in the case the elasticity of substitution of leisure and consumption of goods is higher than the unity we may know that the negative effect of the tax will be higher.

$$\left(1-\ell_t^o\right)\left[\frac{\alpha_o}{\beta_o}u_t^o\right]^{\frac{1}{\sigma}}+u_t^o\left(1-\ell_t^o\right)-u_t^o-\varepsilon=0$$

Take the level of labor demand as constant and then define

$$f\left(u_t^o\right)\equiv\left[\frac{\alpha_o}{\beta_o}u_t^o\right]^{\frac{1}{\sigma}}\ell e_t^o+u_t^o\,\ell e_t^o-u_t^o-\varepsilon=0$$

The level of the wage rate is defined implicitly by $f\,(u_t^0)$=0

We can apply the theorem of the implicit function to $f\,(u_t^0)$=0 to compute $\dfrac{d\,u_t^o}{d\pi}$ and

find out that $\dfrac{d\,u_t^o}{d\pi}\geq 1\Leftrightarrow\dfrac{\sigma}{1-\sigma}\geq 1-\ell e_t^o$

When σ >1 as the left side of the above inequality is negative, then the change in the tax will never be surpassed by the change in the wage rate. In the case σ <1, the result is indeterminate as long as we know the equilibrium value for the employment of the old.

Effects in the general equilibrium model

In our general equilibrium approach we consider the direct and all the indirect effects of the transfer policy. In order to do this we must find the solution of the general equilibrium model.

We return to $S(\ \ell e_t^o\ )=0$, which, under the defined policy instrument becomes

$$J\left(\ell e_t^o\right)=\ell e_t^o$$

Where

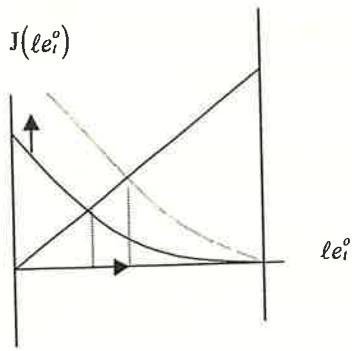
$$J\left(\ell e_t^o\right)\equiv\lambda_1\left(\delta_2\left(1-\ell e_t^o\right)^{1-\delta_1}(1+n)^{\delta_1}+\pi(1+n)\right)\left(1-\ell e_t^o\right)^{\frac{\delta 1}{\sigma}}\quad \text{where } \lambda_1=\left[\frac{\beta_o}{\alpha_o}\frac{1}{\delta_2(1+n)^{\delta_1}}\right]^{\frac{1}{\sigma}}$$

$$J'\left(\ell e_t^o\right)\Big|_{\pi=0}<0$$

$$J(0)=\lambda_1\left(\delta_2(1+n)^{\delta_1}+\pi(1+n)\right)\quad \text{and}\quad J(1)\equiv 0,$$

$\dfrac{\partial J\left(\ell e_t^o\right)}{\partial\pi}>0$ implies an upwards shift of curve $J\left(\ell e_t^o\right)$.

We make a graph of this curve and of the 45 degrees line to find the solution as defined by $J\left(\ell e_t^o\right)=\ell e_t^o$:



An increase in the level of the lump sum transfers increases the level of leisure during the old age: $\frac{d\,\ell e_t^o}{d\pi}>0$, and as $u_t^o=\delta_2\left(\frac{1+n}{\ell_t^o}\right)^{\delta_1}$, the wage rate of the old workers also increases: .

Then, take $\frac{\beta_o}{\alpha_o}\frac{1}{u_t^o}=\left(\frac{\ell e_t^o}{a_t^o}\right)^{\sigma}$, and both $\frac{d\,\ell e_t^o}{d\pi}>0$ and $\frac{d\,u_t^o}{d\pi}>0$ imply that $\frac{d\,a_t^o}{d\pi}>0$.

The utility of the initial generation always improves with lump sum transfers from the young to the old generation

$$W_0=\frac{\alpha_o\left(a_0^o\right)^{1-\sigma}+\beta_o\left(\ell e_0^o\right)^{1-\sigma}}{1-\sigma}\Rightarrow\frac{dW_0}{d\pi}>0,$$

Proposition 1: The utility of the initial generation increases with transfers to the old financed with a lump sum tax. Then, a policy of transfers to the young is not Pareto efficient.

Next, we find out the conditions under which a lump sum tax financed transfers to the old policy increases the level of utility of generations after the initial one.

Before this we obtain an analytical expression for $\frac{d\left(\ell e_t^o\right)}{d\pi}$
The analytical expression for $d\left(\ell e_t^o\right)/d\pi$

We use the theorem of the implicit function on J $\left(\ell e_t^o\right)-\ell e_t^o$ and compute

$$\left.\frac{d\left(\ell e_t^o\right)}{d\pi}\right|_{\pi=0}=\frac{1}{\delta_2}\frac{(1+n)^{1-\delta_1}\,\ell e_t^o\left(1-\ell e_t^o\right)^{\delta_1}}{\ell e_t^o\left[(1-\delta_1)+\frac{\delta_1}{\sigma}\right]+\left(1-\ell e_t^o\right)}$$

where all expressions are evaluated at a zero tax rate.

Necessary and sufficient conditions for $\left.\frac{dW_t}{d\pi}\right|_{\pi=0} \geq 0$

Then, we are ready to obtain the necessary and sufficient conditions for utility improvement under the transfer policy. For this we compute $\left.\frac{dW_t}{d\pi}\right|_{\pi=0}$

After some tedious mathematical manipulation we find out that

$$\left.\frac{dW_t}{d\pi}\right|_{\pi=0} \geq 0 \quad \Leftrightarrow \quad 1-\frac{R_{\pi=0}}{1+n} \geq \ell e_t^o \frac{\delta_1}{\sigma}\left(\frac{R_{\pi=0}}{1+n}-1\right)$$

If $1-\frac{R_{\pi=0}}{1+n} \geq 0$, then $\left.\frac{dW_t}{d\pi}\right|_{\pi=0} \geq 0 \quad \Leftrightarrow \quad 1 \geq -\ell e_t^o \frac{\delta_1}{\sigma}$ holds,

If $1-\frac{R_{\pi=0}}{1+n} < 0$, then $-1 \geq \ell e_t^o \frac{\delta_1}{\sigma}$ does not hold, then $\left.\frac{dW_t}{d\pi}\right|_{\pi=0} < 0$

Therefore, the necessary and sufficient conditions for utility improvement with the increase in the lump sum tax can be summarized in our second proposition

Proposition 2: A policy of transfers from the young to the old generation, financed with a lump sum tax, improves utility of all generations after the initial if and only if the interest rate is lower than the population growth rate

2. Transfers financed with at consumption tax

2.1. Transfers to the young financed by a consumption tax

When transfers to the young generation are being financed with a consumption tax, $\tau_y = \tau_o = \varepsilon = 0$, and then the budget of the government becomes

$$\tau\left(a_t^y(1+n)+a_{t-1}^o\right) = -\pi, \quad \text{with } \pi < 0$$

The solution for ℓe_i^o will be defined from $\mathrm{H}(\ell e_i^o) = 0$

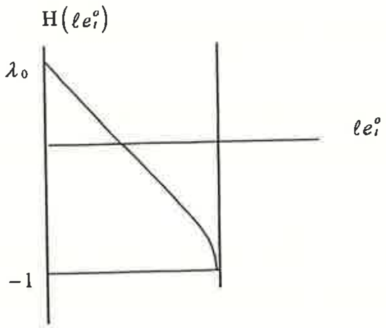
where $\mathrm{H}(\ell e_i^o)$ is defined as $\mathrm{H}(\ell e_i^o) \equiv \lambda_0 (1-\ell e_i^o)^{\frac{(1-\sigma)\delta_1}{\sigma}+1} - \ell e_i^o$

here, $\lambda_0 = (1+\tau)^{\frac{1-\sigma}{\sigma}} \left(\frac{\beta_o}{\alpha_o} \right)^{1/\sigma} (\delta_2)^{\frac{\sigma-1}{\sigma}} (1+n)^{\delta_1 \frac{(\sigma-1)}{\sigma}}$,

$\mathrm{H}(0) = \lambda_0$ and $\mathrm{H}(1) = -1$

$\mathrm{H}'(\ell e_i^o) < 0$ for $0 < \ell e_i^o < 1$, $\mathrm{H}'(0) = -1$ and $\mathrm{H}'(1) \to -\infty$

The graph for function H(ℓe_i^o) would look like the following

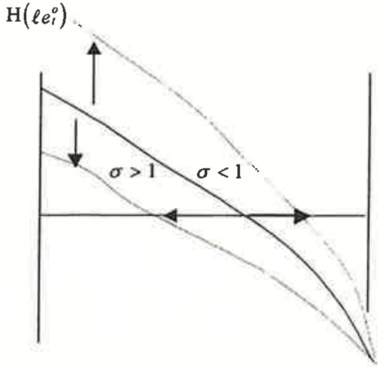


The level of the leisure at equilibrium is lower than the minimum of $\mathrm{H}(\ell e_i^o)$: $\ell e_i^o < \ell e_{\min}^o$.

We compute the effects of a change in the consumption tax rate.

$$\frac{\partial \mathrm{H}(\ell e_i^o)}{\partial \tau} = \frac{1-\sigma}{\sigma} \frac{\ell e_i^o}{1+\tau}$$

Therefore, when $\sigma < 1$, function H(ℓe_i^o) shifts upwards and then $d(\ell e_i^o)/d\tau > 0$, and when $\sigma > 1$, function H(ℓe_i^o) shifts downwards and then $d(\ell e_i^o)/d\tau < 0$.



Effects of a change in on the wage rate of the old workers

$$u_t^o = \delta_2 \left(\frac{1+n}{1-\ell e_t^o} \right)^{\delta_1}$$

$$\frac{d u_t^o}{d\tau} = \delta_1 u_t^o \frac{1}{1-\ell e_t^o} \frac{d \ell e_t^o}{d\tau}$$

$$\text{when} \quad \sigma < 1, \; \frac{d(\ell e_t^o)}{d\tau} > 0, \text{ then } \frac{d u_t^o}{d\tau} > 0$$

$$\text{when} \quad \sigma > 1, \; \frac{d(\ell e_t^o)}{d\tau} < 0, \text{ then } \frac{d u_t^o}{d\tau} < 0$$

The analytical expression of $d(\ell e_t^o)/d\tau$

Applying the theorem of the implicit function to H(ℓe_t^o), we can find an analytical expression for $d(\ell e_t^o)/d\tau$

$$\frac{d(\ell e_t^o)}{d\tau} = \frac{\frac{1-\sigma}{\sigma} \frac{1}{1+\tau} \ell e_t^o}{\frac{\ell e_t^o}{1-\ell e_t^o} \left[\frac{(1-\sigma)\delta_1}{\sigma} + 1 \right] + 1}$$

The optimization problem of the household of the initial generation

The indirect utility function of the initial generation will be the following

$$W_0 = \frac{\alpha_o\left(\frac{\alpha_o}{\beta_o}\frac{1}{1+\tau}u_0^o\right)^{\frac{1-\sigma}{\sigma}} + \beta_o}{1-\sigma}\left(\ell e_0^o\right)^{1-\sigma}$$

where ℓe_0^o is defined by $\mathrm{H}\left(\ell e_0^o\right)=0$, and $u_t^o = \delta_2\left(\frac{1+n}{1-\ell e_t^o}\right)^{\delta_1}$

After some mathematical manipulations which we do not include here, we get

$$\left.\frac{dW_0}{d\tau}\right|_{\tau=0} = \beta_o\left(\ell e_0^o\right)^{-\sigma}\left\{-\frac{1-\ell e_t^o}{\sigma}+\left(\frac{\delta_1}{\sigma}+\frac{1}{\ell e_0^o}\right)\frac{d\,\ell e_0^o}{d\tau}\right\}$$

Using the expression we found for $\frac{d\left(\ell e_t^o\right)}{d\tau}$

$$\left.\frac{dW_0}{d\tau}\right|_{\tau=0} \geq 0 \Leftrightarrow \left(\frac{\delta_1}{\sigma}+\frac{1}{\ell e_0^o}\right)\frac{(1-\sigma)\ell e_0^o}{1+\frac{\ell e_0^o}{\left(1-\ell e_0^o\right)}\left[\frac{(1-\sigma)\delta_1}{\sigma}+1\right]} \geq \left(1-\ell e_0^o\right)$$

From the above, it is immediate that when $\sigma > 1$, $\left.\frac{dW_0}{d\tau}\right|_{\tau=0} \leq 0$.

When, σ <1 develop the above formula to see that

$$\left.\frac{dW_0}{d\tau}\right|_{\tau=0} \geq 0 \Leftrightarrow \delta_1\frac{1-\sigma}{\sigma}\ell e_0^o + (1-\sigma) \geq \left(1-\ell e_0^o\right) + \ell e_0^o\left[\frac{(1-\sigma)\delta_1}{\sigma}+1\right]$$

simplifying even further: $\left.\frac{dW_0}{d\tau}\right|_{\tau=0} \geq 0 \Leftrightarrow (1-\sigma) \geq 1$

Nevertheless, as σ >0, then, $\left.\frac{dW_0}{d\tau}\right|_{\tau=0} \leq 0$

We get our third proposition:

Proposition 3: The utility of the initial generation falls with the increase in the consumption tax. A policy of transfers to the young based in a consumption tax does not produce a Pareto efficient policy.

The volume of transfers

Let's find an expression for the lump sum transfers to the young. As transfers are financed with a consumption tax exclusively, $\tau_y = \tau_o = \varepsilon = 0$ and the budget of the government becomes

$$\tau\left(a_t^y(1+n)+a_{t-1}^o\right)=-\pi(1+n) \text{ with } \pi<0$$

With $a_t^o = \dfrac{u_t^o\left(1-\ell e_t^o\right)}{1+\tau}$ and $a_t^y = \dfrac{u_t^y-\pi}{1+\tau}$, and also using $\dfrac{u_t^y}{u_{t-1}^o}=\dfrac{\delta_1}{\delta_2}\ell_t$

$$\pi=-\frac{\tau}{1+n}u_{t-1}^o\left(1-\ell e_{t-1}^o\right)\left(\frac{\delta_1}{\delta_2}+1\right)$$

We can also find that $\left.\dfrac{d\pi}{d\tau}\right|_{\tau=0}=-u_t^y\left(1+\dfrac{\delta_2}{\delta_1}\right)$

Effect of transfers on the utility of the first and following generations

$$W_t=\frac{\alpha_y\left(\dfrac{u_t^y-\pi}{(1+\tau)}\right)^{1-\sigma}+\alpha_o\left(\dfrac{\alpha_o}{\beta_o}u_t^o\dfrac{1}{(1+\tau)}\right)^{\frac{1-\sigma}{\sigma}}\left(\ell e_t^o\right)^{1-\sigma}+\beta_o\left(\ell e_t^o\right)^{1-\sigma}}{1-\sigma}$$

Effects on the level of consumption of goods in the young age

In order to evaluate the effects of transfers on the utility enjoyed during the youth it is enough to evaluate the sign of . $\dfrac{d a_t^y}{d\tau}$. This is because the volume of leisure is zero in the case we are considering.

We compute and introduce the three expressions , $\left.\dfrac{du_t^y}{d\tau}\right|_{\tau=0}$, $\left.\dfrac{d\pi}{d\tau}\right|_{\tau=0}$ and $\left.u_t^y\right|_{\tau=0}$ in the above equation and get:

$$\left.\frac{da_t^y}{d\tau}\right|_{\tau=0}=\left[\frac{\delta_2}{\delta_1}-\delta_2\frac{1}{1-\ell e_{t-1}^o}\frac{d\,\ell e_{t-1}^o}{d\tau}\right]\left.u_t^y\right|_{\tau=0}$$

Besides, using the expression we found for $\frac{d\left(\ell e_t^o\right)}{d\tau}$, we immediately get $\left.\frac{da_t^y}{d\tau}\right|_{\tau=0} > 0$.

Proposition 4: The utility enjoyed during the youth increases when the consumption tax that finances transfers to the young is raised.

The utility of the first and following generations

The total utility of the household is given by the addition of the utility enjoyed during the youth and the utility enjoyed during the old age: $W_t = U_t^y + U_t^o$

$$\frac{dU_t^y}{d\tau} = \alpha_y\left(a_t^y\right)^{-\sigma}\frac{da_t^y}{d\tau}$$

Or using the expression for $\left.\frac{da_t^y}{d\tau}\right|_{\tau=0}$ we have already computed

$$\left.\frac{dU_t^y}{d\tau}\right|_{\tau=0} = \alpha_y\left(a_t^y\right)^{1-\sigma}\left[\frac{\delta_2}{\delta_1} - \delta_2\frac{1}{1-\ell e_{t-1}^o}\frac{d\,\ell e_{t-1}^o}{d\tau}\right]$$

We can put together the above expression with the expression we already computed for $\left.\frac{dU_t^o}{d\tau}\right|_{\tau=0}$ to get:

$$\left.\frac{dU_t^y}{d\tau}\right|_{\tau=0} + \left.\frac{dU_t^o}{d\tau}\right|_{\tau=0} = \beta_o\left(\ell e_t^o\right)^{-\sigma}\tilde{U}$$

$$\tilde{U} = \frac{\alpha_y}{\beta_o}\left(\frac{a_t^y}{\ell e_t^o}\right)^{-\sigma} a_t^y\left[\frac{\delta_2}{\delta_1} - \delta_2\frac{1}{1-\ell e_t^o}\frac{d\,\ell e_t^o}{d\tau}\right] + \left\{-\frac{1-\ell e_t^o}{\sigma} + \left[\frac{1}{\sigma}\delta_1 + \frac{1}{\ell e_t^o}\right]\frac{d\,\ell e_t^o}{d\tau}\right\}$$

Using the expression we found for $\frac{d\left(\ell e_t^o\right)}{d\tau}$ is easy to see that $\tilde{U} \geq 0 \Leftrightarrow \frac{R|_{\tau=0}}{1+n} \geq 1$

Proposition 5: Transfers to the young with a consumption tax increases the utility of the first and following generations when the interest rate surpasses the population growth rate.

2.2. Transfers to the old with a consumption tax

When transfers to the old are being financed by the consumption tax, $\tau_y = \tau_o = \pi = 0$, and then, the budget of the government reduces to $\tau\,(a_t^y(1+n)+a_{t-1}^o) = \varepsilon$.
As we computed for the case of transfers to the young, we can obtain a final expression

for the transfer parameter . $\varepsilon = \frac{\tau}{\delta_2} u_{t-1}^o\left(1-\ell e_{t-1}^o\right)$.

The solution of the model would be given by $S\left(\ell e_t^o\right)=0$, where

$$S\left(\ell e_t^o\right) = \frac{\beta_o}{\alpha_o}\frac{(1+\tau)^{1-\sigma}}{u_t^o} - \left(\frac{\ell e_t^o}{u_t^o\left(1-\ell e_t^o\right)+\varepsilon}\right)^{\sigma}$$

Using $u_t^o = \delta_2\left(\frac{1+n}{\ell_t^o}\right)^{\delta_1}$ and $\varepsilon = \frac{\tau}{\delta_2} u_t^o\left(1-\ell e_t^o\right)$ we can find a more compact equation,

$\mathrm{M}\left(\ell e_t^o\right)=0$, function of ℓe_t^o exclusively, where $\mathrm{M}\left(\ell e_t^o\right)$

$$\mathrm{M}\left(\ell e_t^o\right) \equiv \lambda_1\left(1-\ell e_t^o\right)^{\frac{1-\sigma}{\sigma}\delta_1+1} - \ell e_t^o$$

here $\lambda_1 = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{\sigma}}\left(\delta_2(1+n)^{\delta_1}\right)^{\frac{\sigma-1}{\sigma}}(1+\tau)^{\frac{1-\sigma}{\sigma}}\left(1+\frac{\tau}{\delta_2}\right)$

Then M( ) defines the solution for . Notice that function M( ) is similar to H( ) (case of transfers to the young) but for the parameter λ_1.

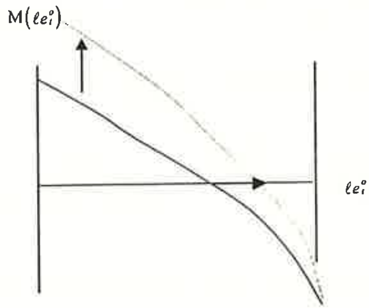
$$\mathrm{M}(0)=\lambda_1,\ \ \mathrm{M}(1)=-1 \text{ and } \mathrm{M}'\left(\ell e_t^o\right)<0$$

$$\frac{\partial \mathrm{M}\left(\ell e_t^o\right)}{\partial \tau} = \left(\frac{1-\sigma}{\sigma}+\frac{1}{\delta_2}\right)\ell e_t^o$$

Applying the theorem of the implicit function to $M(\ell e_i^o)=0$ we find

$$\left.\frac{d(\ell e_i^o)}{d\tau}\right|_{\tau=0}=\frac{\ell e_i^o\left(\frac{1-\sigma}{\sigma}+\frac{1}{\delta_2}\right)}{\frac{\ell e_i^o}{1-\ell e_i^o}\left[\frac{(1-\sigma)\delta_1}{\sigma}+1\right]+1}$$

The graph for function M(ℓe_i^o) will look like in the graph below



As the function shifts upwards with the increase in the tax, the leisure in the old age will increase. The wage rate of the old age workers also increases.

$$\frac{d\,\ell e_i^o}{d\tau}>0\Rightarrow\frac{d\,u_i^o}{d\tau}>0 \qquad \text{because } u_i^o=\delta_2\left(\frac{1}{1-\ell e_i^o}(1+n)\right)^{\delta_1}$$

We also find $\left.\frac{dW_0}{d\tau}\right|_{\tau=0}=\beta_o(\ell e_0^o)^{-\sigma}\left\{-\frac{1-\ell e_i^o}{\sigma}+\left(\frac{\delta_1}{\sigma}+\frac{1}{\ell e_0^o}\right)\left.\frac{d(\ell e_i^o)}{d\tau}\right|_{\tau=0}\right\}$

It can be shown that $\left.\frac{dW_0}{d\tau}\right|_{\tau=0}\geq 0$ holds for any value of the parameters.

Then, we have our proposition 6:

Proposition 6: Utility of the initial generation always improves with transfers to the old financed with a consumption tax.

Effect of transfers on the utility of the first and following generations

Effects on the level of consumption of goods in the young age

In order to evaluate the effects of transfers on the utility enjoyed during the youth it is enough to evaluate the sign of $\frac{d a_t^y}{d\tau}$. This is because in the case we are considering, the volume of leisure is exogenous and does not appear in the utility function.

We have that $a_t^y = \frac{u_t^y}{1+\tau}$ and $u_t^y = \delta_1 \left(\frac{\ell_{t-1}^o}{(1+n)} \right)^{\delta_2}$. The derivatives with respect to the tax parameter are the following

$$\frac{d a_t^y}{d\tau} = \frac{\frac{du_t^y}{d\tau}(1+\tau) - u_t^y}{(1+\tau)^2} \quad \text{, and} \quad \frac{du_t^y}{d\tau} = -\delta_2 \frac{1}{\ell_{t-1}^o} u_t^y \frac{d\ell e_t^o}{d\tau}$$

Therefore, $\left.\frac{d a_t^y}{d\tau}\right|_{\tau=0} = -\left(\delta_2 \frac{d\ell e_t^o}{d\tau} + \ell_{t-1}^o \right) \frac{u_t^y}{\ell_{t-1}^o}$

as $\left.\frac{d(\ell e_t^o)}{d\tau}\right|_{\tau=0} > 0$, $\left.\frac{d a_t^y}{d\tau}\right|_{\tau=0} < 0$

Proposition 7: Utility enjoyed during the youth always falls when transfers to the old are being financed with a consumption tax.

Then, we evaluate the effect of the transfer policy on the total utility.

$$W_t = \frac{\alpha_y (a_t^y)^{1-\sigma}}{1-\sigma} + U_t^o$$

$$\frac{dW_t}{d\tau} = \alpha_y (a_t^y)^{-\sigma} \frac{d a_t^y}{d\tau} + \frac{dU_t^o}{d\tau}$$

The expression for , $\left.\frac{dU_t^o}{d\tau}\right|_{\tau=0}$ is equivalent to the one obtained for the initial generation. Put together all expressions and after reordering and simplifying we find that

$$\frac{dW_t}{d\tau} \geq 0 \Leftrightarrow \frac{R}{1+n} \leq 1$$

Proposition 8: Transfers to the old financed with a consumption tax increases utility of all generations after the initial according to the golden rule.

The case of exogenous labor supply in the old age

When the labor supply in the young age is endogenous and in the old age is endogenous β_y>0, and β_o=0 . In this case, we can use the symmetry properties of the model to find out the effects of intergenerational transfers on the utility of the household.

The basic propositions for this case

Proposition 9: Transfers to the old generation with a consumption tax improves utility of all generations when the interest rate is lower than the population growth rate.

Proposition 10: Transfers to the young based on the consumption tax does not produce a Pareto optimal policy because the utility or the initial generation falls.

The proof is based on the fact that during the old age utility comes only from the consumption of goods. We showed in the previous case that an increase in the consumption tax to finance transfers to the old (young in the present case) does reduce the utility during the youth (old age in the present case).

3. Transfers financed with a labor income tax

3.1. Transfers to the old financed with a labor income tax

In this case, the budget constraint of the government becomes $u_t^y \ell_t^y \tau_y (1+n) = \varepsilon$. Besides, equation $S(\ell e_t^o) = 0$, transforms into

$$\theta_0 (1-\ell e_t^o)^{\frac{(1-\sigma)\delta_1}{\sigma}+1} - \ell e_t^o = 0$$

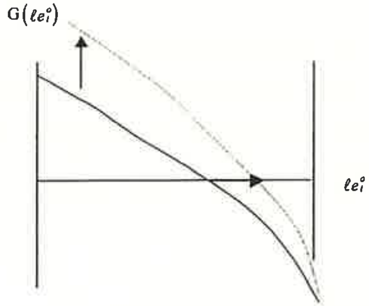
where $\theta_0 = \left[\frac{\beta_o}{\alpha_o}\left(\delta_2 (1+n)^{\delta_1}\right)^{\sigma-1}\right]^{\frac{1}{\sigma}} \left(1 + \frac{\delta_1}{\delta_2}\tau_y\right)$

Then, define $G(\ell e_t^o) \equiv \theta_0 (1-\ell e_t^o)^{\frac{(1-\sigma)\delta_1}{\sigma}+1} - \ell e_t^o$.

Notice the similarity of function G(ℓe_t^o) and function M(ℓe_t^o) we obtained for the case of transfers to the old financed with a consumption tax. The only difference is in the parameter θ_0.

$$\frac{\partial G(\ell e_t^o)}{\partial \tau_y} = \frac{\partial \theta_0}{\partial \tau_y}(1-\ell e_t^o)^{\frac{(1-\sigma)\delta_1}{\sigma}+1} \qquad \left.\frac{\partial \theta_0}{\partial \tau_y}\right|_{\tau_y=0} = \theta_0 \frac{\delta_1}{\delta_2}$$

As the above derivative is positive, curve G(ℓe_t^o) shifts upwards.



showing that leisure in the old age would increase with the increase in the tax.

The analytical expression for $\dfrac{d\,\ell e_i^o}{d\,\tau_y}$

Applying the theorem of the implicit function to G(ℓe_i^o)=0 we get

$$\left.\frac{d\left(\ell e_i^o\right)}{d\tau}\right|_{\tau=0}=\frac{\dfrac{\delta_1}{\delta_2}\ell e_i^o}{\dfrac{\ell e_i^o}{1-\ell e_i^o}\left[\dfrac{(1-\sigma)\delta_1}{\sigma}+1\right]+1}$$

$$\frac{d\,\ell e_i^o}{d\tau}>0\Rightarrow\frac{d\,u_i^o}{d\tau}>0 \qquad \text{because } u_i^o=\delta_2\left(\frac{1}{1-\ell e_i^o}(1+n)\right)^{\delta_1}$$

Using the relation $\dfrac{\beta_o}{\alpha_o}\dfrac{1}{u_i^o}=\left(\dfrac{\ell e_i^o}{a_i^o}\right)^{\sigma}$ we obtained before from the first order conditions for utility maximization, we have that $\dfrac{d\,\ell e_i^o}{d\tau}$ and $\dfrac{d\,u_i^o}{d\tau}>0$ imply that $\dfrac{d\,a_i^o}{d\tau}>0$. Then, we have the following proposition:

Proposition 11: Utility of the household of the initial generation always improves when transfers to the old are financed with a labor income tax on young workers.

Effects on the total utility

Take the indirect utility function of the household

$$W_t = \frac{\alpha_y\left(u_t^y(1-\tau_y)\right)^{1-\sigma} + \alpha_o\left(\frac{\alpha_o}{\beta_o}u_t^o\right)^{\frac{1-\sigma}{\sigma}}\left(\ell e_t^o\right)^{1-\sigma} + \beta_o\left(\ell e_t^o\right)^{1-\sigma}}{1-\sigma}$$

Then, compute $\left.\frac{dW_t}{d\tau_y}\right|_{\tau_y=0}$. After some mathematical developments we do not include

$$\frac{\alpha_y}{\alpha_o}\left(\frac{\delta_2}{\delta_1}(1+n)\right)^{\sigma} = R \quad \text{and} \quad \left.\frac{dW_t}{d\tau_y}\right|_{\tau_y=0} \geq 0 \Leftrightarrow 1 \geq \frac{R_{tax=0}}{1+n}$$

Proposition 12: The utility of all generations after the initial increases with transfers to the old generation financed with a labor income tax on the young, if and only if the interest rate is lower than the population growth rate.

3.2. Transfers to the young financed with a labor income tax

In this case, the budget constraint of the government simplifies as follows:

$$u_t^o \frac{\ell_t^o}{(1+n)} \tau_o = -\pi .$$

Besides, equation S(ℓe_t^o)=0 transforms into

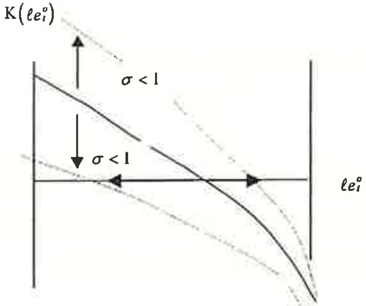
$$\mathrm{K}\left(\ell e_t^o\right) = 0$$

where $\mathrm{K}\left(\ell e_t^o\right) \equiv \theta_0\left(1-\ell e_t^o\right)^{\frac{(1-\sigma)\delta_1}{\sigma}+1} - \ell e_t^o$ and $\theta_1 = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{\sigma}}\left(\delta_2(1+n)^{\delta_1}\right)^{\frac{\sigma-1}{\sigma}}\left(1-\tau_o\right)^{\frac{\sigma-1}{\sigma}}$

The properties of the solution for can be studied in a similar way it has been done for the case of the consumption tax. Notice the similarity of function K(ℓe_i^o) and function G(ℓe_i^o) we obtained for the case of transfers to the old financed with a labor income tax on the young. The only difference is in the parameter θ_o.

$$\frac{\partial K(\ell e_i^o)}{\partial \tau_o} = \frac{1-\sigma}{\sigma}\ell e_i^o$$

Then, curve K(ℓe_i^o) shifts upwards or downwards depending on σ <1 or σ >1.



The analytical expression for $\dfrac{d\,\ell e_i^o}{d\tau}$

$$\left.\frac{d(\ell e_i^o)}{d\tau}\right|_{\tau=0} = (1-\ell e_i^o)\frac{\dfrac{1-\sigma}{\sigma}\ell e_i^o}{\dfrac{(1-\sigma)\delta_1}{\sigma}\ell e_i^o+1}$$

When $\sigma<1 \Rightarrow \dfrac{d\,\ell e_i^o}{d\tau}>0 \Rightarrow \dfrac{d\,u_i^o}{d\tau}>0$ because $u_i^o=\delta_2\left(\dfrac{1}{1-\ell e_i^o}(1+n)\right)^{\delta_1}$

Using the relation $\dfrac{\beta_o}{\alpha_o}\dfrac{1}{u_i^o(1-\tau_0)}=\left(\dfrac{\ell e_i^o}{a_i^o}\right)^{\sigma}$ we obtained from the first order

conditions for utility maximization we have that:

$$a_i^o=\ell e_i^o\left[\frac{\alpha_o}{\beta_o}u_i^o(1-\tau_0)\right]^{\frac{1}{\sigma}} \text{ or using } K(\ell e_i^o)=0$$

$$a_t^o = \delta_2 (1+n)^{\delta_1} (1-\tau_o)(1-\ell e_t^o)^{\delta_2}$$

$$\left.\frac{da_t^o}{d\tau_o}\right|_{\tau_o=0} = -\delta_2 (1+n)^{\delta_1} (1-\ell e_t^o)^{\delta_2} \left[1 + \frac{1}{1-\ell e_t^o} \frac{d\,\ell e_t^o}{d\tau_0} \delta_2\right]$$

$$U_t^o = \frac{\alpha_o (a_t^o)^{1-\sigma} + \beta_o (\ell e_t^o)^{1-\sigma}}{1-\sigma}$$

$$\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} \geq 0 \Leftrightarrow \left.\frac{da_t^o}{d\tau_o}\right|_{\tau_o=0} + u_t^o \left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} \geq 0$$

$$\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} \geq 0 \Leftrightarrow \left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} \delta_1 \geq (1-\ell e_t^o)$$

If we use the expression for $\left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0}$ it is easy to show that $\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} < 0$

Proposition 13: When transfers to the young are being financed with an income tax on the old workers labor income, utility of the initial generation falls.

APPENDIX 3

Intergenerational transfers in a model with endogenous labor supply in both periods

When the labor supply is endogenous in both periods of the household's life, $\beta_y>0$ and $\beta_o>0$, then, the optimization problem of the household takes the form of the problem that appears in the main text.

The first order conditions for utility maximization give the following two equations

$$\frac{\alpha_y}{\beta_y}u_t^y\frac{(1-\tau_y)}{(1+\tau)}=\left(\frac{a_t^y}{\ell e_t^y}\right)^\sigma \quad \text{and} \quad \frac{\beta_o}{\alpha_o}\frac{1}{u_t^o}\frac{(1+\tau)}{(1-\tau_o)}=\left(\frac{\ell e_t^o}{a_t^o}\right)^\sigma$$

The old generation consumes all her endowment, that is,

$$(1-\tau_o)u_t^o(1-\ell e_t^o)+\varepsilon=a_t^o(1+\tau)$$

Then, only the young generation saves and according to the equilibrium condition in the capital market the household also consumes all her endowment during her youth:

$$(1-\tau_y)u_t^y(1-\ell e_t^y)-\pi=a_t^y(1+\tau)$$

Then, using these two expressions in the equations obtained from the first order conditions for maximization we arrive to a system of two equations that determines the solution for ℓe_t^y : ℓe_{t-1}^o

$$S_1(\ell e_t^y,\ell e_{t-1}^o)\equiv\frac{\alpha_y}{\beta_y}u_t^y(1-\tau_y)(1+\tau)^{\sigma-1}-\left(\frac{(1-\tau_y)u_t^y(1-\ell e_t^y)-\pi}{\ell e_t^y}\right)^\sigma \quad \text{and}$$

$$S_2(\ell e_t^y,\ell e_{t-1}^o)\equiv\frac{\beta_o}{\alpha_o}\frac{1}{u_{t-1}^o}\frac{1}{(1-\tau_o)}(1+\tau)^{1-\sigma}-\left(\frac{\ell e_{t-1}^o}{(1-\tau_o)u_{t-1}^o(1-\ell e_{t-1}^o)+\varepsilon}\right)^\sigma$$

and the wage rates are defined from $u_t^y=\delta_1\left(\frac{\ell_{t-1}^o}{\ell_t^y(1+n)}\right)^{\delta_2}$ and $u_t^o=\delta_2\left(\frac{\ell_{t+1}^y}{\ell_t^o}(1+n)\right)^{\delta_1}$.

1. Intergenerational transfers with a consumption tax

Transfers to the young with a consumption tax

When transfers to the young are financed with a consumption tax, $\tau_y = \tau_o = \tau = 0$ and then the budget of the government becomes

$$\tau\left(a_t^y(1+n)+a_{t-1}^o\right)=-\pi(1+n) \text{ with } 0<\tau<1 \text{ and } \pi<0.$$

We can develop this relation and find

$$\pi=-\frac{1}{\delta_2}\frac{\tau}{1+n}u_{t-1}^o\left(1-\ell e_{t-1}^o\right)$$

The solution of the model is given by $S_1\left(\ell e_t^y, \ell e_{t-1}^o\right)=0$ and $S_2\left(\ell e_{t+1}^y, \ell e_t^o\right)=0$. From equation $S_2\left(\ell e_t^y, \ell e_{t-1}^o\right)$ can find a relation between the leisure of the young and the old workers of the following type:

$$\ell e_t^y\left(\ell e_t^o\right)=1-\lambda\,\ell e_{t-1}^o\left(\frac{1-\ell e_{t-1}^o}{\ell e_{t-1}^o}\right)^{\frac{\sigma}{(1-\sigma)\delta_1}+1} \qquad \text{where} \quad \lambda=\left[\frac{(1+\tau)\left(\dfrac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)}}}{\delta_2(1+n)^{\delta_1}}\right]^{\frac{1}{\delta_1}}$$

Features of function $\ell e_t^y\left(\ell e_{t-1}^o\right)$

$$\begin{cases}\sigma<1 \Rightarrow \lim\limits_{\ell e_{t-1}^o \to 0} \ell e_t^y \to -\infty \\ \sigma>1 \Rightarrow \lim\limits_{\ell e_{t-1}^o \to 0} \ell e_t^y \to 1\end{cases} \qquad \begin{cases}\sigma<1 \Rightarrow \lim\limits_{\ell e_{t-1}^o \to 1} \ell e_t^y \to 1 \\ \sigma>1 \Rightarrow \lim\limits_{\ell e_{t-1}^o \to 1} \ell e_t^y \to -\infty\end{cases}$$

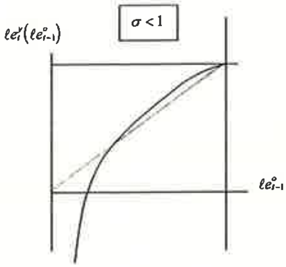
$$\ell e_t^{y\prime}\left(\ell e_{t-1}^o\right)=\frac{1-\ell e_t^y}{\left(1-\ell e_{t-1}^o\right)}\left\{1-\frac{\sigma}{\ell e_{t-1}^o(\sigma-1)\delta_1}\right\}$$

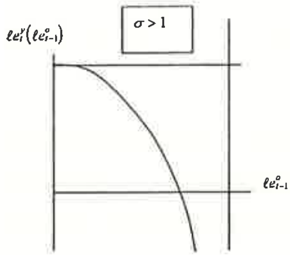
By simple inspection, if , $\sigma<1$, then, $\ell e_t^{y\prime}(\ell e_{t-1}^o)>0$. Besides, if $\sigma>1$, $\ell e_t^{y\prime}(\ell e_{t-1}^o)<0$

because $\frac{\sigma}{(\sigma-1)\delta_1}=\frac{1}{\left(1-\frac{1}{\sigma}\right)\delta_1}>1$, and in the interval $0<\ell e_t^y(\ell e_{t-1}^o)<1$, $\frac{1}{\ell e_{t-1}^o}>1$ too.

$$\ell e_t^{y\prime}(\sigma<1,\ell e_{t-1}^o=1)=0$$

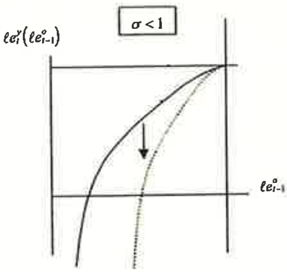
$$\ell e_t^{y\prime}(0,\sigma>1)=0$$

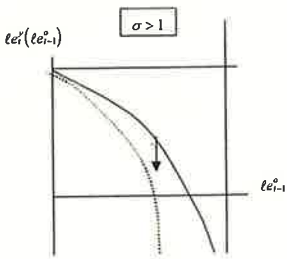




Next, we compute $\frac{\partial \ell e_t^y}{\partial \tau}$, that is, the shift of $\ell e_t^y(\ell e_{t-1}^o)$ due to the change in the consumption tax rate.

$$\frac{\partial \ell e_t^y}{\partial \tau}=-\frac{1}{\delta_1}\frac{1-\ell e_t^y}{1+\tau}$$





Solving the model

Using $S_1\left(\ell e_t^y, \ell e_{t-1}^o\right)=0$ and $\ell e_t^y\left(\ell e_{t-1}^o\right)$, we find and expression $Q\left(\ell e_{t-1}^o\right)=0$ that defines the solution for ℓe_{t-1}^o.

$$Q\left(\ell e_{t-1}^o\right)=\ell e_t^y\left(\ell e_{t-1}^o\right)-\frac{1}{\Phi^{1/\sigma}}\ell e_{t-1}^o\left(\frac{1-\ell e_{t-1}^o}{\ell e_{t-1}^o}\right)^{\frac{1}{(1-\sigma)\delta_1}}$$

Here, $\Phi=\frac{\psi}{\lambda^{(\sigma-1)\delta_1+1}}$, $\psi=\frac{\alpha_y}{\beta_y}\frac{\delta_1(1+\tau)^{\sigma-1}}{\left(\delta_2(1+n)^{\delta_1}\right)^{\sigma}(1+n)^{\delta_2}\mu}$, $\mu=\left(\frac{1}{\delta_2}\right)^{\sigma}\left(\frac{\delta_1}{1+n}+\frac{\tau}{1+n}\right)^{\sigma}$ and

$$\lambda=\left[\frac{(1+\tau)\left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)}}}{\delta_2(1+n)^{\delta_1}}\right]^{\frac{1}{\delta_1}}.$$

Features of $Q\left(\ell e_{t-1}^o\right)$

$$\begin{cases} if \quad \sigma<1 \Rightarrow \lim\limits_{\ell e_{t-1}^o\to 0} Q(0)=-\infty \\ if \quad \sigma>1 \Rightarrow \lim\limits_{\ell e_{t-1}^o\to 0} Q(0)=1 \end{cases} \quad \text{and} \quad \begin{cases} if \quad \sigma<1 \Rightarrow \lim\limits_{\ell e_{t-1}^o\to 0} Q(0)=1 \\ if \quad \sigma>1 \Rightarrow \lim\limits_{\ell e_{t-1}^o\to 0} Q(0)=-\infty \end{cases}$$

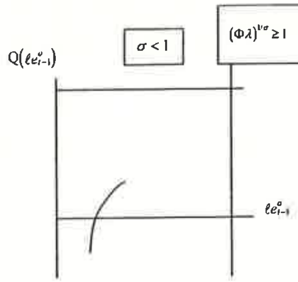
Actually, bounds are determined by the zeros of function $\ell e_t^y\left(\ell e_t^o\right)$.

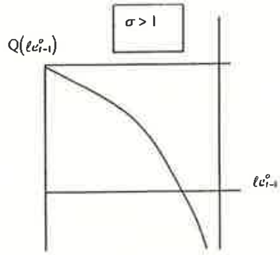
$$\lim_{\ell e_{t-1}^o\to \ell e_{zero}^o} Q=-\frac{1}{\Phi^{1/\sigma}}\ell e_{zero}^o\left(\frac{1-\ell e_{zero}^o}{\ell e_{zero}^o}\right)^{\frac{1}{(1-\sigma)\delta_1}}$$

The derivative

$$Q'\left(\ell e_{t-1}^o\right)=\ell e_t^{y\prime}-\frac{\ell e_t^y}{\ell e_{t-1}^o}\left[1-\frac{1}{(1-\sigma)\delta_1}\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right]$$

When $\sigma > 1$, $\ell e_i^{y\prime}\left(\ell e_{i-1}^o\right) < 0$, then, $Q'\left(\ell e_{i-1}^o\right) < 0$ too, because the expression inside [ ] brackets is positive. When σ <1, it can be shown that, if there is only one solution for the derivative is positive at that solution.

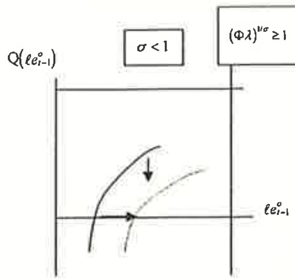


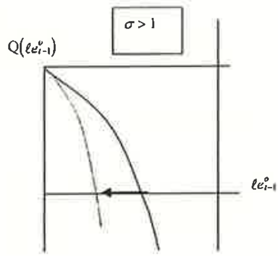


$$\frac{\partial Q}{\partial \tau} = \frac{\partial \ell e_i^y}{\partial \tau} + \frac{1}{\sigma}\Phi^{-1/\sigma-1}\frac{\partial \Phi}{\partial \tau}\ell e_{i-1}^o\left(\frac{1-\ell e_{i-1}^o}{\ell e_{i-1}^o}\right)^{\frac{1}{(1-\sigma)\delta_1}}$$

Therefore, with $\left.\frac{\partial \ell e_i^y}{\partial \tau}\right|_{\tau=0} = -\frac{1-\ell e_i^y}{\delta_1}$ and $\left.\frac{\partial \Phi}{\partial \tau}\right|_{\tau=0} = -\Phi\big|_{\tau=0}\frac{(\sigma+1)}{\delta_1}$, both negative

$\frac{\partial Q}{\partial \tau} < 0$,then $\left.\frac{d\ell e_i^o}{d\tau}\right|_{\tau=0} > 0 \Leftrightarrow \sigma < 1$





Therefore, when $\sigma < 1$, $\left.\frac{d\ell e_i^o}{d\tau}\right|_{\tau=0} > 0$ and when $\sigma > 1$ $\left.\frac{d\ell e_i^o}{d\tau}\right|_{\tau=0} < 0$

Effects of transfers on the utility of the initial generation

The indirect utility function of the initial generation is the following:

$$U_t^o = \beta_o \frac{\left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{\sigma}}\left(u_t^o\frac{1}{(1+\tau)}\right)^{\frac{1-\sigma}{\sigma}}\left(\ell e_t^o\right)^{1-\sigma}+\left(\ell e_t^o\right)^{1-\sigma}}{1-\sigma} \quad \text{with} \quad t=0$$

If we compute the derivative with respect to the consumption tax, and after after several manipulations, we can get

$$\frac{dU_t^o}{d\tau}=\frac{\beta_o}{\left(\ell e_t^o\right)^{1+\sigma}}\frac{\sigma}{\sigma-1}\frac{d\ell e_t^o}{d\tau}$$

As $\left.\frac{d\ell e_t^o}{d\tau}\right|_{\tau=0}>0$ when $\sigma<1$, and $\left.\frac{d\ell e_t^o}{d\tau}\right|_{\tau=0}<0$ when $\sigma>1$, $\frac{dU_t^o}{d\tau}<0$ in any case.

Proposition 14: In a model of endogenous labor supply in the young and old age, the level of utility of the initial generation falls when transfers to the young generation is being financed with a consumption tax.

Effects on the level of utility of the first and following generations

Total utility is given by $W_t=U_t^y+U_t^o$ for t=1,2 ……

$\frac{dU_t^o}{d\tau}$ has already been computed when evaluating the effects on the initial generation.

$$\frac{dU_i^y}{d\tau} = \eta_0\left[\ell e_{i-1}^o\right]^{-\sigma}\left(1-\ell e_{i-1}^o\right)\frac{1+\frac{\eta_1}{\eta_0}\left(\frac{1-\ell e_{i-1}^o}{\ell e_{i-1}^o}\right)^{\frac{1}{\delta_1}-1}}{1-\sigma}\tilde{\mathrm{U}}$$

$$\eta_0 = \alpha_y \frac{\beta_o}{\alpha_o}\mu^{\frac{(1-\sigma)}{\sigma}} \qquad\qquad \eta_1 = \beta_y \Phi^{\frac{\sigma-1}{\sigma}}$$

$$\tilde{\mathrm{U}}\Big|_{\tau=0} = (1-\sigma)\frac{1}{\delta_1} - \left[\frac{\sigma}{\ell e_{i-1}^o}+1\right]\frac{d\,\ell e_{i-1}^o}{d\tau} + \frac{\left(\frac{1-\ell e_{i-1}^o}{\ell e_{i-1}^o}\right)^{\frac{1}{\delta_1}-1}}{1+\frac{\eta_1}{\eta_0}\left(\frac{1-\ell e_{i-1}^o}{\ell e_{i-1}^o}\right)^{\frac{1}{\delta_1}-1}}\frac{\eta_1}{\eta_0}\left\{\frac{1}{\delta_1}\frac{\sigma-1}{\sigma} - \frac{1}{1-\ell e_{i-1}^o}\frac{1}{\ell e_{i-1}^o}\left(\frac{1}{\delta_1}-1\right)\right\}$$

therefore, $\left.\frac{dU_i^y}{d\tau}\right|_{\tau=0} \geq 0 \Leftrightarrow \frac{\tilde{\mathrm{U}}\big|_{\tau=0}}{1-\sigma} \geq 0$.

More specific conditions could be obtained. For example, when

$$\sigma > 1,\ \left.\frac{dU_i^y}{d\tau}\right|_{\tau=0} \geq 0 \Leftrightarrow \tilde{\mathrm{U}}\Big|_{\tau=0} \leq 0.$$

Then notice that when σ >1, all summation terms are positive but the third. Then, for example, take the expression in { } brackets of the third summation term

$$\frac{1}{\delta_1}\frac{\sigma-1}{\sigma} - \frac{1}{1-\ell e_{i-1}^o}\frac{1}{\ell e_{i-1}^o}\left(\frac{1}{\delta_1}-1\right) = \left\{\left(1-\frac{1}{\sigma}\right)\left(1-\ell e_{i-1}^o\right)\ell e_{i-1}^o - (1-\delta_1)\right\}\frac{1}{\delta_1}\frac{1}{1-\ell e_{i-1}^o}\frac{1}{\ell e_{i-1}^o}$$

$$\left(1-\frac{1}{\sigma}\right)\left(1-\ell e_{i-1}^o\right)\ell e_{i-1}^o - (1-\delta_1) < \left(1-\frac{1}{\sigma}\right) - (1-\delta_1) = \delta_1 - \frac{1}{\sigma}$$

Then, for example $\delta_1 < \frac{1}{\sigma}$ is a sufficient condition for $\tilde{\mathrm{U}}\Big|_{\tau=0} \leq 0$.

2. Intergenerational transfers with a labor income tax

2.1. Transfers to the old with a labor income tax on the young workers

The budget constraint of the government becomes . $u_t^y \ell_t^y \tau_y(1+n) = \varepsilon$.

The variables and are determined in the following system of two equations

$$\begin{cases} S_1\left(\ell e_t^y, \ell e_{t-1}^o\right) \\ S_2\left(\ell e_t^y, \ell e_{t-1}^o\right) \end{cases}$$

From the above two equations we can find an equation $Q\left(\ell e_{t-1}^o\right) = 0$ that determines the solution for ℓe_{t-1}^o

$$Q\left(\ell e_{t-1}^o\right) \equiv \frac{1}{c_0} \frac{c_2\left(\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}+\frac{\sigma}{(1-\sigma)\delta_1}}}{c_2\left(\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}}+1} - \left(1-\ell e_{t-1}^o\right)$$

where $c_0 = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)\delta_1}} \left(1+\frac{\delta_1}{\delta_2}\tau_y\right)^{\frac{\sigma}{(1-\sigma)\delta_1}} \frac{1}{(1+n)(\delta_2)^{\frac{1}{\delta_1}}}$,

$$c_1 = \left(\frac{\alpha_y}{\beta_y}\right)^{\frac{1}{(1-\sigma)\delta_2}} \frac{(1-\tau_y)^{\frac{1}{\delta_2}}}{1+n} (\delta_1)^{\frac{1}{\delta_2}} \text{ and } c_2 = \left(\frac{c_1}{c_0}\right)^{\frac{1-\sigma}{\sigma}\delta_2}$$

Besides, $\ell e_t^y = \dfrac{1}{c_2\left(\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}}+1}$

Observe the similarities with $T\left(\ell e_{t-1}^o\right)$ of the case of transfers to the young with a tax on young workers. The graph for $Q\left(\ell e_{t-1}^o\right)$ is similar to the one for $T\left(\ell e_{t-1}^o\right)$.

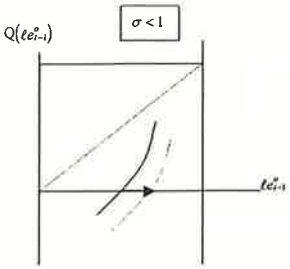
Effects of the change in the tax rate on ℓe^o_{t-1}

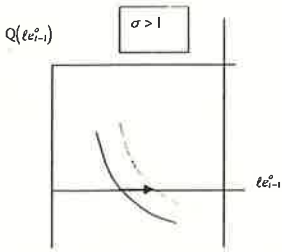
We compute the shift of function $Q\left(\ell e^o_{t-1}\right)$ due to the change in the tax rate τ_y: $\frac{\partial Q}{\partial \tau_y}$.

$$\frac{\partial Q}{\partial \tau_y}=\left(\frac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{1}{\delta_1}\frac{\sigma}{(1-\sigma)}+\frac{\delta_2}{\delta_1}}\frac{1}{c_0}\frac{1}{c_2\left(\frac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{\delta_2}{\delta_1}}+1}q$$

$$q\big|_{\tau_y=0}=-c_2\left\{\frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2}+\frac{1}{\sigma}\ell e^y_t\right\}$$

$$\sigma<1\Rightarrow q\big|_{\tau_y=0}<0\Rightarrow\frac{\partial Q}{\partial \tau_y}<0\Rightarrow\frac{d\,\ell e^o_t}{d\tau_o}>0 \quad \text{and} \quad \sigma>1\Rightarrow q\big|_{\tau_y=0}>0\Rightarrow\frac{\partial Q}{\partial \tau_y}>0\Rightarrow\frac{d\,\ell e^o_t}{d\tau_o}>0$$





The analytical expression for $\frac{d\,\ell e^o_t}{d\tau_y}$

Applying the theorem of the implicit function to $Q\left(\ell e^o_{t-1}\right)=0$

$$\frac{d\,\ell e^o_t}{d\tau_y}=\frac{\left(1-\ell e^o_{t-1}\right)\left\{\frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2}+\frac{1}{\sigma}\ell e^y_t\right\}}{1+\frac{1}{\ell e^o_{t-1}}\frac{\delta_2}{\delta_1}\left\{\frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2}+\ell e^y_t\right\}}$$

Effects on the consumption of goods in the old age: $\frac{d\,a^o_t}{d\tau_y}$

From the first order condition we had $a_t^o = \left(\frac{\alpha_o}{\beta_o}\delta_2(1+n)^{\delta_1}\right)^{1/\sigma}\left(\frac{1-\ell e_{t+1}^y}{1-\ell e_t^o}\right)^{\frac{\delta_1}{\sigma}} \ell e_t^o$

We get a final expression for a_t^0

$$a_t^o = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)}}\left(1+\frac{\delta_1}{\delta_2}\tau_y\right)^{\frac{1}{(1-\sigma)}}\left(\frac{1-\ell e_{t-1}^o}{\ell e_{t-1}^o}\right)^{\frac{1}{(1-\sigma)}} \ell e_t^o$$

$$\frac{d a_t^o}{d\tau_y} = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)}}\left(1+\frac{\delta_1}{\delta_2}\tau_y\right)^{\frac{1}{(1-\sigma)}}\left(\frac{1-\ell e_{t-1}^o}{\ell e_{t-1}^o}\right)^{\frac{1}{(1-\sigma)}} \ell e_t^o \left\{\frac{1}{1-\sigma}\frac{\delta_1}{\delta_2}+\frac{1}{\ell e_t^o}\left[1-\frac{1}{1-\sigma}\frac{1}{(1-\ell e_{t-1}^o)}\right]\frac{d\,\ell e_t^o}{d\tau_y}\right\}$$

Effects on the utility of the initial generation

Take the indirect utility function of the initial generation

$$W_t = U_t^o = \frac{\alpha_o(a_t^o)^{1-\sigma}+\beta_o(\ell e_t^o)^{1-\sigma}}{1-\sigma} \quad \text{with} \quad t=0$$

then, compute

$$\left.\frac{dU_t^o}{d\tau_y}\right|_{\tau_y=0} = \alpha_o(a_t^o)^{-\sigma}\left\{\left.\frac{da_t^o}{d\tau_y}\right|_{\tau_y=0} + u_t^o\left.\frac{d\,\ell e_t^o}{d\tau_y}\right|_{\tau_y=0}\right\}$$

Using $u_t^o = \delta_2\left(\frac{\ell_{t+1}^y}{\ell_t^o}(1+n)\right)^{\delta_1}$ we can write $\left.\frac{dU_t^o}{d\tau_y}\right|_{\tau_y=0} = \alpha_o(a_t^o)^{-\sigma}\mathrm{H}$

$$\mathrm{H} = \left(\frac{\beta_o}{\alpha_o}\right)^{\frac{1}{(1-\sigma)}}\left(\frac{1-\ell e_t^o}{\ell e_t^o}\right)^{\frac{1}{(1-\sigma)}}\frac{\ell e_t^o}{1-\sigma}\frac{\delta_1}{\delta_2}\left\{\frac{\ell e_t^o\frac{\delta_1}{\delta_2}+\frac{\sigma}{\delta_2}}{\ell e_t^o\frac{\delta_1}{\delta_2}+\frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2}+\ell e_{t+1}^y}\right\}$$

When, $\sigma<1 \Rightarrow \mathrm{H}>0 \Rightarrow \left.\frac{dU_t^o}{d\tau_y}\right|_{\tau_y=0} >0$ can be directly inferred from simple inspection

of the above expression. When σ >1 we have the same result because of the following:

$$\ell e_t^o \delta_1 + \delta_2 \ell e_{t+1}^y + \frac{\sigma}{(1-\sigma)} < 0 \text{ because } \ell e_t^o \delta_1 + \delta_2 \ell e_{t+1}^y < 1 \text{ and } \frac{\sigma}{(1-\sigma)} < -1$$

Then, if $\sigma > 1 \Rightarrow \mathrm{H} > 0 \Rightarrow \left.\frac{dU_t^o}{d\tau_y}\right|_{\tau_y=0} > 0$

Proposition 15: In a model of endogenous labor supply in both periods, the youth and the old age, a policy of transfers to the old based on the labor income tax increases the level of utility of the initial generation in any case.

Effects of the change in the tax on the consumption of goods of the young

$$a_t^y = \left[\frac{\alpha_y}{\beta_y}\right]^{\frac{1}{(1-\sigma)}} \left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}} \ell e_t^y$$

$$\frac{d a_t^y}{d\tau_y} = \left[\frac{\alpha_y}{\beta_y}\right]^{\frac{1}{(1-\alpha)}} \left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}} \left[-\frac{1}{(1-\sigma)}\frac{1}{1-\ell e_t^y}\left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}} + 1\right] \frac{d\,\ell e_t^y}{d\tau_y}$$

For $\sigma > 1$, $\frac{d a_t^y}{d\tau_y} > 0$ can be notice by direct inspection of the above expression.

For $\sigma < 1$, $\frac{d a_t^y}{d\tau_y} > 0 \Leftrightarrow$

$$\frac{1}{(1-\sigma)}\frac{1}{1-\ell e_t^y}\left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}} < 1 \Leftrightarrow \left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}} < (1-\ell e_t^y)(1-\sigma) \Leftrightarrow 1-(\ell e_t^y)^{\frac{1}{\sigma}}(1-\sigma)^{\frac{1-\sigma}{\sigma}} < \ell e_t^y$$

Effects of the change in the tax on the leisure of the young

As we found before $\ell e_t^y = \dfrac{1}{c_2\left(\dfrac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}} + 1}$

$$\frac{d\,\ell e_t^y}{d\tau_y} = -\left[\ell e_t^y\right]^2 c_2 \left[\frac{\delta_2}{\delta_1}\frac{1}{\ell e_{t-1}^o(1-\ell e_{t-1}^o)}\frac{d\,\ell e_{t-1}^o}{d\tau_y} - \frac{1}{\sigma}\right]\left(\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}}$$

Effect on the utility in the youth can be found using the expressions for $\frac{d a_t^y}{d\tau_y}$ and $\frac{d\ell e_t^y}{d\tau_y}$

$$\left.\frac{dU_t^y}{d\tau_y}\right|_{\tau_y=0} = \alpha_y \left(a_t^y\right)^{-\sigma} \mathrm{U}$$

$$\mathrm{U} = \left.\frac{d a_t^y}{d\tau_y}\right|_{\tau_y=0} + \delta_1 \left(\frac{\ell_{t-1}^o}{\ell_t^y(1+n)}\right)^{\delta_2} \left.\frac{d\ell e_t^y}{d\tau_y}\right|_{\tau_y=0}, \text{ or else}$$

$$\mathrm{U} \equiv \left(\frac{1-\ell e_{t-1}^o}{1-\ell e_t^y}\right)^{\delta_2} \delta_1 \left(\frac{1}{1+n}\right)^{\delta_2} \left(\frac{1-\ell e_t^y}{\ell e_t^y}\right) \frac{1}{1-\ell e_t^y} \left\{1 - \frac{1}{(1-\sigma)}\left(\frac{1-\ell e_t^y}{\ell e_t^y}\right)^{\frac{1}{(1-\sigma)}}\right\}$$

$$\frac{d\ell e_t^y}{d\tau_y} = -\left[\ell e_t^y\right]^2 c_2 \left[\frac{\frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2} + \frac{1}{\sigma}\ell e_t^y}{\ell e_{t-1}^o \frac{\delta_1}{\delta_2} + \frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2} + \ell e_t^y} - \frac{1}{\sigma}\right] \left(\frac{\ell e_{t-1}^o}{1-\ell e_{t-1}^o}\right)^{\frac{\delta_2}{\delta_1}}$$

if $\frac{d\ell e_t^o}{d\tau_o} > 0$, $\ell e_{t-1}^o \frac{\delta_1}{\delta_2} + \frac{\sigma}{(1-\sigma)}\frac{1}{\delta_2} + \ell e_t^y \geq 0$, therefore $\frac{d\ell e_t^y}{d\tau_y} \geq 0$

$$\left.\frac{dU_t^y}{d\tau_y}\right|_{\tau_y=0} \geq 0 \Leftrightarrow \frac{d\ell e_t^y}{d\tau_y}\mathrm{U}$$

then notice that for $\sigma > 1$, $\mathrm{U} > 0$ and as $\frac{d\ell e_t^y}{d\tau_y} \geq 0$, $\left.\frac{dU_t^y}{d\tau_y}\right|_{\tau_y=0} \geq 0$

Conditions for improvement of the total utility are less easy to compute we have left this development for future work.

2.2. Transfers to the young with a labor income tax on the old workers

We show that transfers to the young based on the labor income tax does not produce a Pareto efficient policy because the initial generation sees her level of utility reduced.

The budget of the government becomes $u^o_{t-1}\frac{\ell^o_{t-1}}{1+n}\tau_o = -\pi$ with $\pi < 0$

The solution for the variable ℓe^o_{t-1} can be found solving the equation $\mathrm{T}\left(\ell e^o_{t-1}\right) = 0$ where

$$\mathrm{T}\left(\ell e^o_{t-1}\right) \equiv \frac{1}{b_0}\frac{b_2\left(\dfrac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{1}{\delta_1}\frac{\sigma}{(1-\sigma)}+\frac{\delta_2}{\delta_1}}}{b_2\left(\dfrac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{\delta_2}{\delta_1}}+1} - \left(1-\ell e^o_{t-1}\right)$$

Besides, we have $\ell e^y_t = \dfrac{1}{b_2\left(\dfrac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{\delta_2}{\delta_1}}+1}$ with $b_2 = \left(b_1 b_0\right)^{\frac{\sigma-1}{\sigma}\delta_2}$

Features of $\mathrm{T}\left(\ell e^o_{t-1}\right)$

if $\sigma < 1$, $\lim_{\ell e^o_{t-1}\to 0}\mathrm{T}\left(\ell e^o_{t-1}\right) = -1$

if $\sigma > 1$, $\lim_{\ell e^o_{t-1}\to 0}\mathrm{T}\left(\ell e^o_{t-1}\right) = +\infty$

If $\sigma < 1$, $\lim_{\ell e^o_{t-1}\to 1}\mathrm{T}\left(\ell e^o_{t-1}\right) = +\infty$.

If $\sigma > 1$, $\lim_{\ell e^o_{t-1}\to 1}\mathrm{T}\left(\ell e^o_{t-1}\right) = 0$.

The derivative

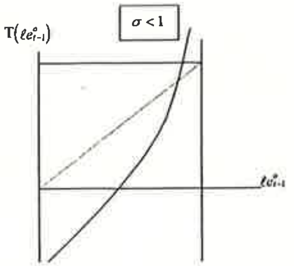
$$\mathrm{T}'\left(\ell e^o_{t-1}\right) = 1 + \frac{b_2}{b_0}\left[b_2\left(\frac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{\delta_2}{\delta_1}}+1\right]^{-2}\{\ \}$$

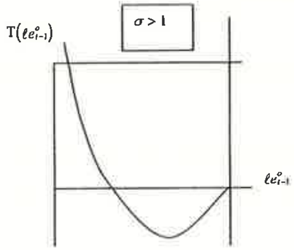
$$\{\ \} = \left(\frac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{1}{\delta_1}\frac{\sigma}{(1-\sigma)}+\frac{\delta_2}{\delta_1}}\frac{1}{\ell e^o_{t-1}\left(1-\ell e^o_{t-1}\right)}\frac{1}{\delta_1}\left\{\frac{\sigma}{(1-\sigma)}+\delta_2+\left[\frac{\sigma}{(1-\sigma)}-\delta_1\right]b_2\delta_2\left(\frac{\ell e^o_{t-1}}{1-\ell e^o_{t-1}}\right)^{\frac{\delta_2}{\delta_1}}\right\}$$

When $\sigma < 1$,

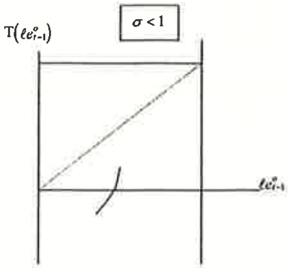
If also $\dfrac{\sigma}{(1-\sigma)} - \delta_1 = \dfrac{1}{\left(\dfrac{1}{\sigma}-1\right)} - \delta_1 > 0$, then, we can be sure that $\mathrm{T}'\left(\ell e^o_{t-1}\right) > 0$

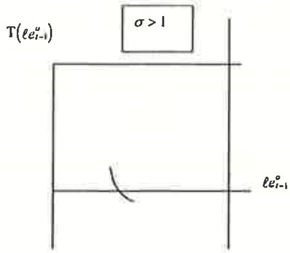
When there is only one solution, the graph for T(ℓe^o_{t-1}) will look like the following





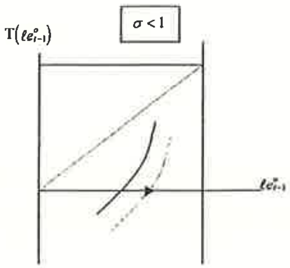
n the case there is more than one solution, the graph of $T(\ell e^o_{t-1})$ at the neighborhood of at least one of the solutions would be

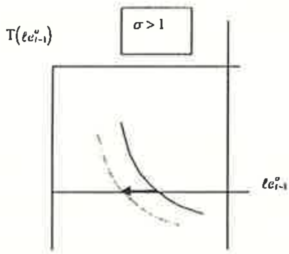




Effects of the change in the tax rate

It can be shown that $\dfrac{\partial T}{\partial \tau_o} < 0$ in any case.





If $\sigma < 1 \Rightarrow \dfrac{d\,\ell e^o_t}{d\tau_o} > 0$

If $\sigma > 1 \Rightarrow \dfrac{d\,\ell e^o_t}{d\tau_o} < 0$

Computing $\frac{d a_i^o}{d\tau_o}$

From the first order condition we had $a_i^o = \left(\frac{\alpha_o}{\beta_o} u_i^o (1-\tau_o)\right)^{1/\sigma} \ell e_i^o$

After introducing the expression for the wage rate in the old age and other simplifications, we get a final expression for a_i^o

$$a_i^o = \left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{(\sigma-1)}} \left(\frac{\ell e_i^o}{1-\ell e_i^o}\right)^{\frac{1}{\sigma-1}} \ell e_i^o$$

$$\frac{d a_i^o}{d\tau_o} = \left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{(\sigma-1)}} \left(\frac{\ell e_i^o}{1-\ell e_i^o}\right)^{\frac{1}{\sigma-1}} \left(\frac{1}{\left(1-\ell e_i^o\right)} \frac{1}{\sigma-1} + 1\right) \frac{d \ell e_i^o}{d\tau_o}$$

If $\sigma < 1 \Rightarrow \frac{d \ell e_i^o}{d\tau_o} > 0$, then $\frac{d a_i^o}{d\tau_o} < 0$. This is because the first summation term in the expression between brackets

$\left|\frac{1}{\sigma-1} \frac{1}{1-\ell e_{i-1}^o}\right| > 1$ and negative.

If $\sigma > 1$, $\sigma > 1 \Rightarrow \frac{d \ell e_i^o}{d\tau_o} < 0$, then $\frac{d a_i^o}{d\tau_o} < 0$

Finally $\frac{d a_i^o}{d\tau_o} < 0$ in any case.

While $\frac{d a_i^o}{d\tau_o} < 0$, $\frac{d \ell e_i^o}{d\tau_o}$, could be positive or negative depending on σ then, to investigate the effect of the increase in the tax on the utility of the initial generation we study the indirect utility function directly.

Effect of an increase of the old labor income tax on the utility of the initial generation

$$\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} = \alpha_o\left(a_t^o\right)^{-\sigma}\left\{\left.\frac{da_t^o}{d\tau_o}\right|_{\tau_o=0} + \delta_2\left(\frac{\ell_{t+1}^y}{\ell_t^o}(1+n)\right)^{\delta_1}\left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0}\right\}$$

As we computed before

$$\frac{da_t^o}{d\tau_o} = \left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{(\sigma-1)}}\left(\frac{\ell e_t^o}{1-\ell e_t^o}\right)^{\frac{1}{\sigma-1}}\left[\frac{1}{\left(1-\ell e_t^o\right)}\frac{1}{\sigma-1}+1\right]\frac{d\ell e_t^o}{d\tau_o}$$

$$\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} \geq 0 \Leftrightarrow \ddot{U}\left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} \geq 0$$

$$\ddot{U} \equiv \left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{(\sigma-1)}}\left(\frac{\ell e_t^o}{1-\ell e_t^o}\right)^{\frac{1}{\sigma-1}}\left\{1+\frac{\ell e_t^o}{1-\ell e_t^o}-\frac{1}{\left(1-\ell e_t^o\right)}\frac{1}{1-\sigma}\right\}$$

When, $\sigma > 1$, $\dddot{U} > 0$

$$\left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} = \left(\frac{\alpha_o}{\beta_o}\right)^{\frac{1}{(\sigma-1)}}\left(\frac{\ell e_t^o}{1-\ell e_t^o}\right)^{\frac{1}{\sigma-1}}\ddot{U}\left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} < 0 \quad \text{but as} \quad \left.\frac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} < 0,\ \left.\frac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} < 0$$

When, $\sigma < 1$,

$$\left\{1-\ell e_t^o+\ell e_t^o-\frac{1}{1-\sigma}\right\}\frac{1}{\left(1-\ell e_t^o\right)} = \left\{1-\frac{1}{1-\sigma}\right\}\frac{1}{\left(1-\ell e_t^o\right)}$$

$1-\dfrac{1}{1-\sigma} = -\dfrac{\sigma}{1-\sigma} < 0$, then, $\ddot{U} < 0$

When $\sigma < 1$, $\ddot{U} < 0$ and $\left.\dfrac{d\ell e_t^o}{d\tau_o}\right|_{\tau_o=0} < 0$, imply $\left.\dfrac{dU_t^o}{d\tau_o}\right|_{\tau_o=0} < 0$

Proposition 16: In a model of endogenous labor supply for the youth and old age, a policy of transfers to the young based on the labor income tax does not produce a Pareto efficient policy because the initial generation sees her level of utility reduced.