# A Note on Semantic Tools for Modal Logics

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*Abstract*: Two kinds of semantical tools for interpreting propositional modal formulas are introduced, and their close relation to each other is explained in this note. Although it contains no novel findings about modal logics, all the facts and their proofs in this note are extremely significant to students and researchers in this field of studies for future use.

# 1 Introduction

There are two prominent ways to investigate mathematical logics: the syntactical one and the semantical one. The former is suitable for showing some properties of *each particular logic*, by defining an equivalent syntactical system to it, and by utilizing the mathematical induction on the definition of syntactical objects tactically. On the other hand, the latter has an advantage in establishing some *general results*, like of the form: "all logics that are determined by some semantical objects with such and such conditions have these good properties." ([3], [5], [7],[8], [9])

Semantical systems are used for attaching a meaning to each formula in a given logic, determining a set of *true* formulas in some setting of the system, and also determining a set of *valid* formulas, which may hopefully be equal to the set of *theorems* of the given logic.

For interpreting propositional modal formulas, there are two main different semantical systems: *modal algebras* and *general frames*. Both systems can interpret propositional modal formulas in their own way, and can give *completeness theorems* to some classes of modal logics. The class of logics which can be determined by modal algebras and the class which can be determined by general frames are a little different, but these two types of systems are useful in investigating propositional modal logics and it can be said that both serve us the same mathematical objects to interpret modal formulas in a certain extent.

There exists a close relation between modal algebras and general frames. For example, for a given modal algebra, there exists a general frame which corresponds to the original algebra, in a sense that both semantics make the same set of formulas valid. On the other hand, for a given general frame, there is a modal algebra which corresponds to that frame. These facts are a part of the well-known *representation theory* of modal algebras.

In this note, some useful theorems in the representation theory of modal algebras are explained systematically. After introduction of modal logics in the second section, modal algebras, and general frames, the ways how to construct frames from algebras, and conversely, how to construct frames from algebras are discussed in section 3. Moreover, it is presented in this section that homomorphisms between two algebras correspond to p-morphisms between two frames.

*Craig's Interpolation Property* and *Halldén Completeness* are two major examples of syntactical properties of mathematical logics. To each property, an equivalent algebraic condition and an equivalent frame-theoretic condition are already known. In section 4, the equivalence of conditions of algebras and frames for both properties are presented on the basis of representation theory.

Several syntactical methods are established for proving a given logic to possess the Craig's Interpolation Property. One of them is known as *Inseparable Tableaux Method* using semantic tableaux. In section 5, this method is rewritten in an algebraically equivalent style in order to apply it to a broader class of modal logics.

In the last section, outlook for this research area in the future are discussed with some open questions.

# 2 Preliminaries

First of all, several syntactical notions are introduced to define propositional normal modal logics. The propositional modal language $\mathcal{L}$ consists of the following set of symbols: (1) a countable set of propositional variables $\{p_0, p_1, \cdots\}$, (2) a set of connectives $\{\bot, \wedge, \neg, \Box\}$, and (3) a pair of parentheses $\{(,)\}$. The set $\Phi(:= \Phi(\mathcal{L}))$ of modal formulas in the language $\mathcal{L}$ is defined in a usual way. The following connectives and formulas using them are introduced as abbreviations: $\top := \neg\bot$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\Diamond\varphi := \neg(\Box(\neg\varphi))$. A *normal modal logic* in the language $\mathcal{L}$ is defined as a subset $\mathbf{L}$ of $\Phi$ that contains: (1) all classical tautologies of $\Phi$, and (2) a formula of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and is closed under (3) the *Uniform Substitution* ($\varphi \in \mathbf{L}$ implies $\varphi[\psi/p_i] \in \mathbf{L}$), (4) the *Modus Ponens* ($\varphi, \varphi \to \psi \in \mathbf{L}$ implies $\psi \in \mathbf{L}$), and (5) the *Necessitation* ($\varphi \in \mathbf{L}$ implies $\Box\varphi \in \mathbf{L}$).

The smallest normal modal logic of $\Phi$ is denoted by $\mathbf{K}$. A formula $\varphi$ is called a *theorem* of a logic $\mathbf{L}$ if $\varphi \in \mathbf{L}$. For two normal logics $\mathbf{L}$ and $\mathbf{L}'$, $\mathbf{L}'$ is a *normal extension* of $\mathbf{L}$ if $\mathbf{L} \subseteq \mathbf{L}'$. For a normal logic $\mathbf{L}$ and a set $\Sigma$ of formulas, the smallest normal extension of $\mathbf{L}$ containing also $\Sigma$ is denoted by $\mathbf{L} \oplus \Sigma$. The class of all normal extensions of $\mathbf{L}$ is denoted by $\textsc{Next}(\mathbf{L})$.

One of the main figures of this note, that is, modal algebra, is defined in the following.

**Definition 2.1 (Modal algebras)** A *modal algebra* is a structure $\mathfrak{A} := \langle A, \cap, \cup, -, I, 0, 1\rangle$, where $\langle A, \cap, \cup, -, 0, 1\rangle$ is a boolean algebra, and $I$ is a unary operator satisfying: for any $a, b \in A$, (1) $I(1) = 1$ and (2) $I(a \cap b) = I(a) \cap I(b)$. ∎

To interpret each formula on a modal algebra $\mathfrak{A}$, an assignment function, or *valuation* $v : \Phi \to A$ is used. For each variable $p_i$, $v$ assigns an element in $A$ to the variable, that is, $v(p_i) \in A$. For formulas in general, the assignment by $v$ is defined in the following inductive way: $v(\bot) = 0$, $v(\neg\varphi) = -v(\varphi)$, $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$, and $v(\Box\varphi) = I(v(\varphi))$. A formula $\varphi$ is *true* in a *model* $\langle \mathfrak{A}, v\rangle$, if $v(\varphi) = 1$ in $\mathfrak{A}$. A formula $\varphi$ is *valid* in $\mathfrak{A}$ ($\mathfrak{A} \models \varphi$), if for any valuation $v : \Phi \to A$, $\varphi$ is true in $\langle \mathfrak{A}, v\rangle$. For a class $\mathcal{C}$ of modal algebras, a formula $\varphi$ is *valid* in $\mathcal{C}$ ($\mathcal{C} \models \varphi$), if for any algebra $\mathfrak{A} \in \mathcal{C}$, $\varphi$ is valid in $\mathfrak{A}$.

Let $\mathcal{C}_{\mathbf{K}}$ be the class of all modal algebras. For the normal modal logic $\mathbf{K}$, the following *completeness theorem* holds: for any formula $\varphi$, $\varphi$ is a theorem of $\mathbf{K}$ if and only if $\varphi$ is valid in $\mathcal{C}_{\mathbf{K}}$. This theorem is also expressed as: *the logic* $\mathbf{K}$ *is complete with respect to the class of all modal algebras.* Similarly, for any normal modal logic $\mathbf{L}$, there exists a suitable class $\mathcal{C}_{\mathbf{L}}$ of modal algebras such that $\mathbf{L}$ is complete with respect to $\mathcal{C}_{\mathbf{L}}$.

For any class $\mathcal{C}$ of modal algebras, the set of formulas $\mathbf{L}(\mathcal{C}) := \{\varphi \in \Phi \mid \mathfrak{A} \models \varphi$ for any $\mathfrak{A} \in \mathcal{C}\}$ determines a normal logic. On the other hand, for any normal modal logic $\mathbf{L}$, the class of modal algebras $\mathcal{V}(\mathbf{L}) := \{\mathfrak{A} \in \mathcal{K} \mid \mathfrak{A} \models \varphi$ for each $\varphi \in \mathbf{L}\}$ turns out to be a special class of algebras called a variety. A *variety* is a class of algebras which is defined by a set of identities. A famous theorem for characterizing varieties by G. Birkhoff ([1]) is that a class $\mathcal{C}$ of algebras is a variety if and only if $\mathcal{C}$ is closed under (1) taking homomorphic images, (2) taking subalgebras, and (3) taking direct products. For those reasons, every variety corresponds to a normal modal logic.

The other figure of this note is frame, which is defined in the following.

**Definition 2.2 (General frames)** A (*general*) *frame* is a structure $\mathcal{F} := \langle W, R, P \rangle$, where $W$ is a non-empty set of *worlds*, $R$ is a binary relation on $W$, and $P$ is a subset of $\mathcal{P}(W)$ that contains $\emptyset$ and $W$, and is closed under the set-theoretic intersection, the set-theoretic complement, and the operation $I_R$ defined as: $I_R(X) := \{x \in W \mid \forall x \in W(xRy$ implies $y \in X)\}$ for all $X \in \mathcal{P}(W)$. ∎

On a frame, a little different type of assignment function is used for interpreting formulas. A *valuation* $V$ on a frame $\mathcal{F} := \langle W, R, P \rangle$ is a function from a set of propositional variables to some members in $P$, that is, $V(p_j) \in P$ for a variable $p_j$. When a valuation $V$ is fixed, a meaning of a formula in a model $\mathfrak{M} := \langle \mathcal{F}, V \rangle$ is assigned in the following. A formula $\varphi$ is *true* at a point $x \in W$ in a model $\mathfrak{M}$ ($\mathfrak{M} \models_x \varphi$ in symbol) is defined inductively as:

(0) $\mathfrak{M} \not\models_x \bot$ always holds.

(1) $\mathfrak{M} \models_x p_i$ if and only if $x \in V(p_i)$.

(2) $\mathfrak{M} \models_x \neg\psi$ if and only if $\mathfrak{M} \not\models_x \psi$.

(3) $\mathfrak{M} \models_x \psi \wedge \eta$ if and only if $\mathfrak{M} \models_x \psi$ and $\mathfrak{M} \models_x \eta$.

(4) $\mathfrak{M} \models_x \Box\psi$ if and only if $\forall y \in W, \big(xRy$ implies $\mathfrak{M} \models_y \psi\big)$.

A formula $\varphi$ is *valid* in a frame $\mathcal{F}$ ($\mathcal{F} \models \varphi$), if $\langle \mathcal{F}, V \rangle \models_x \varphi$ for any valuation $V$ on $\mathcal{F}$ and for any point $x \in W$. For a class $\mathcal{D}$ of frames, a formula $\varphi$ is *valid* in $\mathcal{D}$ ($\mathcal{D} \models \varphi$), if $\mathcal{F} \models \varphi$ for any $\mathcal{F} \in \mathcal{D}$.

As is discussed in the next section, modal algebras and general frames are categorically *dual* to each other, and so, the similar completeness theorem also holds for general frames: for every normal modal logic $\mathbf{L}$, there exists a suitable class $\mathcal{D}$ of general frames such that $\mathbf{L}$ is complete with respect to $\mathcal{D}$.

A *Kripke* frame is a special sort of general frame $\mathcal{F} := \langle W, R, P \rangle$, where $P = \mathcal{P}(W)$, and it is denoted only by $\mathcal{F} := \langle W, R \rangle$. A logic $\mathbf{L}$ is *Kripke complete* if it is complete with respect to a class of Kripke frames. It is not the case that every normal modal logic is complete with respect to some class of Kripke frames. In fact, quite a few normal modal logics are shown to be Kripke incomplete ([21], [6], [2], [22], [18]). Other terminologies for propositional modal logics follow the usage in Chagrov and Zakharyashev's book [4].

# 3 Duality between modal algebras and general frames

## 3.1 Jónsson-Tarski duality between modal algebras and general frames

A deep connection exists between the class of all modal algebras and a subclass of the class of all general frames, that is now called *Jónsson-Tarski Duality* ([13], [14]). This means that, for a given modal algebra $\mathfrak{A}$, a general frame can be constructed from this $\mathfrak{A}$, and that validates exactly the same set of formulas as $\mathfrak{A}$ does. Conversely, a modal algebra can be also constructed from a given frame $\mathcal{F}$ which validates the same set of formulas as the frame $\mathcal{F}$ does. This good correspondence between modal algebras and general frames is explained in this section, together with a correspondence between *morphisms* of two classes of such *objects*.

Now, morphisms for algebras and frames are defined here.

**Definition 3.1 (Homomorphisms of modal algebras)** Let $\mathfrak{A}, \mathfrak{B}$ be modal algebras.

(1) $f : A \to B$ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ if,

(a) $f(1_A) = 1_B$,

(b) $f(a \cap_A b) = f(a) \cap_B f(b)$,

(c) $f(-_A a) = -_B f(a)$,

(d) $f(I_A(a)) = I_B(f(a))$.

(2) $f : A \to B$ is an *embedding*, if it is a homomorphism, and also one to one.

∎

**Definition 3.2 (Frame-morphisms of general frames)** Let $\mathcal{F} = \langle W, R, P\rangle$ and, $\mathcal{G} = \langle U, S, Q\rangle$ be general frames.

(1) $g : W \to U$ is a *frame-morphism* from $\mathcal{F}$ to $\mathcal{G}$ if,

(a) $\forall x, y \in W\big({}_xR_y$ implies ${}_{g(x)}S_{g(y)}\big)$,

(b) $\forall x \in W, \forall a \in U\big({}_{g(x)}S_a$ implies $\exists z \in W(g(z) = a$ and ${}_xR_z)$,

(c) $\forall X \in Q, g^{-1}(X) \in P$.

(2) $g : W \to U$ is a *p-morphism*, if it is a frame-morphism, and also onto.

∎

Let $\mathfrak{A} = \langle A, \cap, \cup, -, 0, 1\rangle$ be a boolean algebra. A subset $F$ of $A$ is called a *filter*, if $F$ satisfies: (1) $1 \in F$, and for $x, y \in A$, (2) $x, y \in F$ implies $x \cap y \in F$ and (3) $x \in F$ and $x \leq y$ implies $y \in F$. A filter $F$ is *proper*, if $0 \notin F$. A proper filter $F$ is *prime*, if $x \cup y \in F$ implies $x \in F$ or $y \in F$ for $x, y \in A$. A prime filter $F$ in $\mathfrak{A}$ is a maximal proper filter in $\mathfrak{A}$. As its dual, a subset $J$ of $A$ is called an *ideal*, if $J$ satisfies: (1) $0 \in J$, and for $x, y \in A$, (2) $x, y \in J$ implies $x \cup y \in J$, and (3) $x \in J$ and $y \leq x$ implies $y \in J$. An ideal $J$ is *proper*, if $1 \notin J$. A proper ideal $J$ is *prime*, if $x \cap y \in J$ implies $x \in J$ or $y \in J$ for $x, y \in A$. A prime ideal $J$ in $\mathfrak{A}$ is a maximal proper ideal in $\mathfrak{A}$.

One of the most important lemmas about prime filters is the following:

**Lemma 3.3** Let $\mathfrak{A} := \langle A, \cap, \cup, -, 0, 1\rangle$ be a boolean algebra, $F \subseteq A$ a proper filter in $\mathfrak{A}$, and $a \in A$ such that $a \notin F$. Then there exists a prime filter $G$ satisfying $F \subseteq G$ and $a \notin G$.

**Proof** : It is easily proved by a standard use of *Zorn's lemma*. □

The transformation from an algebra into a frame, and their connection are shown in the following.

**Proposition 3.4** Let $\mathfrak{A} = \langle A, \cap, \cup, -, I, 0, 1\rangle$ be a modal algebra. Define $\mathfrak{A}_* := \langle F_p(A), R_{\mathfrak{A}}, P_{\mathfrak{A}}\rangle$ as follows: $F_p(A)$ is the set of all prime filters in $\mathfrak{A}$, $R_{\mathfrak{A}}$ is a binary relation on $F_p(A)$ defined as: ${}_F R_{\mathfrak{A}G}$ if and only if $\forall a \in A, \big( I(a) \in F$ implies $a \in G\big)$, and $P_{\mathfrak{A}} := \{\theta(a) \mid a \in A\}$, where $\theta(a) := \{F \in F_p(A) \mid a \in F\}$. Then $\mathfrak{A}_*$ is a general frame with the following property: $\forall \varphi \in \Phi\big(\mathfrak{A}_* \models \varphi$ if and only if $\mathfrak{A} \models \varphi\big)$.

**Proof** : In order to be checked that $P_{\mathfrak{A}}$ is closed under some needed operations, it will be seen that $\theta(x)$ is an embedding from $\mathfrak{A}$ into $\langle P_{\mathfrak{A}}, \cap, \cup, -, I_R, \emptyset, F_p(A)\rangle$. Suppose $a \not\leq b$ in $\mathfrak{A}$. Then there is a prime filter $F \in F_p(A)$ such that $a \in F$ and $b \notin F$. This means that $\theta(a) \not\subseteq \theta(b)$. Thus $\theta$ is one to one. Easy calculation shows that $\theta(-a) = -\theta(a)$ and $\theta(a \cap b) = \theta(a) \cap \theta(b)$. On the modal operator, suppose $F \in \theta(I(a))$. Then $I(a) \in F$. Take any prime filter $G$ such that ${}_F R_{\mathfrak{A}G}$ and then, of course, $a \in G$, which means that $G \in \theta(a)$. Therefore $F \in I_R(\theta(a))$. Hence $\theta(I(a)) \subseteq I_R(\theta(a))$. Conversely suppose $F \notin \theta(I(a))$ which means that $I(a) \notin F$. Put $G := \{b \in A \mid I(b) \in F\}$. Then, $a \notin G$, and it is easy to see that $G$ is a proper filter. Therefore by Lemma 3.3, there exists a prime filter $G'$ such that $G \subseteq G'$ and $a \notin G'$. This $G'$ satisfies ${}_F R_{\mathfrak{A}G'}$ and $a \notin G'$, which implies that $F \notin I_R(\theta(a))$. Thus $I_R(\theta(a)) \subseteq \theta(I(a))$ is proved. Hence $P_{\mathfrak{A}}$ is closed under those operations, and so, $\mathfrak{A}_*$ is a general frame.

For a valuation $v$ on $\mathfrak{A}$ and a valuation $V$ on $\mathfrak{A}_*$, suppose $V(p_i) = \theta(v(p_i))$. Here, denote the set $\{F \in F_p(A) \mid \langle \mathfrak{A}_*, V\rangle \models \varphi\}$ by $V(\varphi)$ for any formula $\varphi$ built from $p_i$'s, then it is immediately seen by induction that $V(\varphi) = \theta(v(\varphi))$. Therefore $V(\varphi) = F_p(A)$ in $\mathfrak{A}_*$ if and only if $v(\varphi) = 1$ in $\mathfrak{A}$. Thus $\mathfrak{A}_* \models \varphi$ if and only if $\mathfrak{A} \models \varphi$ holds for any formula $\varphi$. □

The converse transformation from a frame into an algebra also exists.

**Proposition 3.5** Let $\mathcal{F} = \langle W, R, P\rangle$ be a general frame. Define $\mathcal{F}^* := \langle P, \cap, \cup, -, I_R, \emptyset, W\rangle$. Then $\mathcal{F}^*$ is a modal algebra with the following property: $\forall\varphi \in \Phi(\mathcal{F}^* \models \varphi$ if and only if $\mathcal{F} \models \varphi)$.

**Proof** : It is easy to be checked that $I_R(W) = W$ and $I_R(X \cap Y) = I_R(X) \cap I_R(Y)$ for all $X, Y \in P$, and so $\mathcal{F}^*$ is a modal algebra. For a valuation $V$ on $\mathcal{F}$ and a valuation $v$ on $\mathcal{F}^*$, suppose that $V(p_i) = v(p_i) \in P$. Denote $V(\varphi) := \{x \in W \mid \langle \mathcal{F}, V\rangle \models_x \varphi\}$ for any formula $\varphi$ which is constructed from $p_i$'s, then it is immediate to be seen by induction that $V(\varphi) = v(\varphi)$. Therefore, it is the case that $V(\varphi) = W$ if and only if $v(\varphi) = W$. Hence $\mathcal{F}^* \models \varphi$ if and only if $\mathcal{F} \models \varphi$ holds for any formula $\varphi$. □

Note that about the composition of two kinds of transformation $(\cdot)^*$ and $(\cdot)_*$ introduced above, $(\mathfrak{A}_*)^*$ is isomorphic to $\mathfrak{A}$ itself for any modal algebra $\mathfrak{A}$, whereas, for general frames, $(\mathcal{F}^*)_*$ is not always isomorphic to $\mathcal{F}$. A general frame $\mathcal{F}$ is called *descriptive*, if $(\mathcal{F}^*)_*$ is isomorphic to $\mathcal{F}$. ($(\mathcal{F}^*)_* \cong \mathcal{F}$ in symbol) For this reason, it is the class of descriptive general frames that exactly corresponds to the class of all modal algebras from a categorical viewpoint. Characterization of descriptive frames is presented later in this section.

There is also a good correspondence between morphisms of algebras and frames.

**Lemma 3.6** Let $\mathfrak{A}_1, \mathfrak{A}_2$ be modal algebras.

(1) Suppose $f : A_1 \to A_2$ is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. Then, a map $g : \mathfrak{A}_{2*} \to \mathfrak{A}_{1*}$ defined as: $g(F) := \{a \in A_1 \mid f(a) \in F\}$ is a frame-morphism.

(2) In (1), if $f$ is also one to one (i.e. an embedding), then $g$ is onto

(i.e. a p-morphism).

**Proof** : (1): The first thing to check is that $g(F)$ is a prime filter in $\mathfrak{A}_1$ for any prime filter $F$ in $\mathfrak{A}_2$. $0 \notin g(F)$ because $f(0) = 0 \notin F$. Suppose $a \in g(F)$ and $a \leq b$. Then $f(a) \in F$ and $f(a) \leq f(b) \in F$, and so, $b \in g(F)$. Suppose $a, b \in g(F)$. Then $f(a), f(b) \in F$, that implies that $f(a) \cap f(b) = f(a \cap b) \in F$, and so, $a \cap b \in g(F)$. Suppose $a \cup b \in g(F)$ and $a \notin g(F)$. Then, $f(a \cup b) = f(a) \cup f(b) \in F$. Since $F$ is a prime filter, either $f(a) \in F$ or $f(b) \in F$. Here, if the former holds, then $a \in g(F)$ contradicts to the assumption. Thus, the latter holds, and so, $b \in g(F)$. Hence $g(F)$ is indeed a prime filter in $\mathfrak{A}_1$.

Denote $\mathfrak{A}_{1*} := \langle W_1, R_1, P_1 \rangle$ and $\mathfrak{A}_{2*} := \langle W_2, R_2, P_2 \rangle$. For $F, G \in W_2$, suppose ${}_F R_{2G}$. Take any $I(a) \in g(F)$. Because $f(I(a)) = I(f(a)) \in F$, and ${}_F R_{2G}$, $f(a) \in G$, and so, $a \in g(G)$, which means that ${}_{g(F)} R_{1g(G)}$.

For $F \in W_2$ and $G \in W_1$, suppose ${}_{g(F)} R_{1G}$. Put $Y := \{a \in A_2 | I(a) \in F\}$, $Z := \{a \in A_2 | f(b) \leq a \text{ for some } b \in G\}$, and $H' := \{a \in A_2 | c \cap d \leq a \text{ for some } c \in Y \text{ and } d \in Z\}$.

**Claim:** This $H'$ is a proper filter.

It is easily seen that $1 \in H'$. Suppose that $0 \in H'$. Then, by its definition $c \cap d \leq 0$ holds in $\mathfrak{A}_2$ for some $c \in Y$ and some $d \in Z$. Since $\mathfrak{A}_2$ is a boolean algebra, $d \leq -c$ follows. Then, by the fact that $d \in Z$, $f(b) \leq d \leq -c$ also holds for some $b \in G$. Therefore, $c = -(-c) \leq -f(b) = f(-b)$, and so, $I(c) \leq I(f(-b)) = f(I(-b))$ follows. Here, since $c \in Y$, $I(c) \in F$, which implies that $f(I(-b)) \in F$. Then $I(-b) \in g(F)$ by the definition of the map $g$. Now ${}_{g(F)} R_{1G}$ forces that $-b \in G$, that leads to a contradiction. Hence it is proved that $0 \notin H'$.

Suppose $a \in H'$ and $a \leq b$ in $\mathfrak{A}_2$. Then $c \cap d \leq a \leq b$ for some $c \in Y$ and some $d \in Z$. Therefore $b \in H'$. Suppose $a, b \in H'$ in $\mathfrak{A}_2$. Then, $c \cap d \leq a$ for some $c \in Y$ and some $d \in Z$, and $c' \cap d' \leq b$ for some $c' \in Y$

and some $d' \in Z$. Therefore $(c \cap d) \cap (c' \cap d') = (c \cap c') \cap (d \cap d') \leq a \cap b$ holds for $c \cap c' \in Y$ and $d \cap d' \in Z$. Thus $a \cap b \in H'$ holds in $\mathfrak{A}_2$. The claim has just been proved.

Therefore, by Lemma 3.3, there exists a prime filter $H$ such that $H' \subseteq H$. Now, suppose $I(a) \in F$ for an arbitrary $a \in A_2$. Then, of course $a \cap 1 \leq a$ for $a \in Y$ and $1 \in Z$, which implies that $a \in H' \subseteq H$. Hence ${}_F R_{2H}$ follows. Here, it is the case that $g(H) = G$, because for any $a \in G$, $f(a) \leq f(a)$ holds, of course, but it means that $f(a) \in Z$. By the fact that $1 \cap f(a) \leq f(a)$, $f(a) \in H' \subseteq H$ follows, which implies that $G \subseteq g(H)$. Since $G$ is a prime filter, it must be a maximal proper filter, and so, $g(H) = G$ is proved.

Finally it is checked the correspondence between $P_{\mathfrak{A}_1}$ and $P_{\mathfrak{A}_2}$. For any $X \in P_{\mathfrak{A}_1} = \{\theta_1(a) \mid a \in A_1\}$, $X = \theta_1(a)$ for some $a \in A_1$, and $F \in g^{-1}(\theta_1(a))$ if and only if $g(F) \in \theta_1(a)$ if and only if $a \in g(F)$ if and only if $f(a) \in F$ if and only if $F \in \theta_2(f(a))$. This means that $g^{-1}(X) = g^{-1}(\theta_1(a)) = \theta_2(f(a)) \in P_{\mathfrak{A}_2} = \{\theta_2(b) \mid b \in A_2\}$.

(2): Suppose that $f$ is one to one. Consider any $F \in F_p(A_1)$. Put $G' := \{a \in A_2 \mid f(b) \leq a \text{ for some } b \in F\}$.

**Claim:** This $G'$ is a proper filter.

Suppose $0 \in G'$. Then by the definition of $G'$, there is $b \in F$ such that $f(b) \leq 0$, that implies $f(b) = 0 = f(0)$. Since $f$ is one to one, $b = 0 \in F$, but this is a contradiction.

Suppose $a \in G'$ and $a \leq b$ in $\mathfrak{A}_{2*}$. Then, $f(c) \leq a \leq b$ for some $c \in F$. Hence $b \in G'$. Furthermore suppose $a, b \in G'$. Then, $f(c) \leq a$ holds for some $c \in F$, and $f(d) \leq b$ holds for some $d \in F$. Now, $f(c \cap d) \leq f(c) \cap f(d) \leq a \cap b$ follows, and so, $a \cap b \in G'$ since $c \cap d \in F$. Thus $G'$ is indeed a proper filter.

Therefore by Lemma 3.3, there exists a prime filter $G$ in $\mathfrak{A}_2$ such that

$G' \subseteq G$. This $G$ satisfies that $g(G) = F$, because, for $a \in F$, a trivial fact $f(a) \leq f(a)$ means that $f(a) \in G' \subseteq G$, and so, $a \in g(G)$. On the other hand, for $b \notin F$, it is the case that $-b \in F$, which implies by the same reasoning above that $-b \in g(G)$, that is $b \notin g(G)$. Hence it is proved that $g$ is onto. □

Moreover, a homomorphism between duals of frames can be constructed out of a frame-morphism between original frames.

**Lemma 3.7** Let $\mathcal{F}_1 = \langle W_1, R_1, P_1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2, P_2 \rangle$ be general frames.

(1) Suppose $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a frame-morphism. Then, a map $g : \mathcal{F}_2{}^* \to \mathcal{F}_1{}^*$ defined as: $g(X) := \{x \in W_1 \mid f(x) \in X\}$ for $X \in P_2$ is a homomorphism from $\mathcal{F}_2{}^*$ to $\mathcal{F}_1{}^*$.

(2) In (1), if $f$ is also onto (i.e. p-morphism), then $g$ is one to one (i.e. an embedding).

**Proof** : (1): Since $f$ is a frame-morphism, for any $X \in P_2$, $f^{-1}(X) = g(X) \in P_1$ holds. Thus $g$ is well-defined. Then the conservation of each operator of the modal algebras has only to be checked. It is trivial that $g(W_2) = W_1$. It is also easy to be seen that $g(X \cap Y) = g(X) \cap g(Y)$, $g(-X) = -g(X)$. On the modal operator, suppose $x \in g(I_{R_2}(X))$, which is equivalent to $f(x) \in I_{R_2}(X)$. Take any $y \in W_1$ such that $xR_1y$. Then $f(x)Rf(y)$ holds, and so, $f(y) \in X$. Hence $y \in g(X)$, which implies that $x \in I_{R_2}(g(X))$. Conversely, suppose $x \in I_{R_1}(g(X))$ and take any $z \in W_2$ such that $f(x)R_2z$. Then there is some $u \in W_1$ such that $xR_1u$ and $f(u) = z$. Since $x \in I_{R_1}(g(X))$, $u \in g(X)$, and so, $f(u) = z \in X$. Therefore $f(x) \in I_{R_2}(X)$, that implies $x \in g(I_{R_2}(X))$. Thus $g(I_{R_2}(X)) = I_{R_1}(g(X))$ follows. It is proved that $g$ is a homomorphism.

(2): Suppose $g(X) = g(Y)$ for $X, Y \in P_2$. Here, consider any $x \in X$. Since $f$ is onto, there exists some $y \in W_1$ such that $f(y) = x \in X$. Thus $y \in g(X) = g(Y)$ holds, and so, $f(y) = x \in Y$. Hence $X \subseteq Y$ follows. □

So far, it is proved that the class of modal algebras correspond to the class of general frames, and vice a versa, by dual transformations $(\cdot)^*$ and $(\cdot)_*$. However, this correspondence is, to say exactly, the class of all modal algebras to the class of all descriptive general frames, which will be seen in the next subsection.

### 3.2 Characterization of descriptive frames

A descriptive general frame is a frame whose bidual transformation is isomorphic to itself. The class of all descriptive frames is characterized by a subclass of frames which possess the following three frame properties. In the beginning, the three properties of general frames are defined below.

**Definition 3.8** Let $\mathcal{F} = \langle W, R, P \rangle$ be a general frame.

(1) $\mathcal{F}$ is *differentiated*, if for all $x, y \in W$, $x = y$ if and only if $\forall X \in P\big(x \in X$ is equivalent to $y \in X\big)$,

(2) $\mathcal{F}$ is *tight*, if for all $x, y \in W$, $xRy$ if and only if $\forall X \in P\big(x \in I_R(X)$ implies $y \in X\big)$,

(3) $\mathcal{F}$ is *compact*, if for any $\chi \subseteq P$, that $\chi$ has the finite intersection property implies $\bigcap \chi \neq \emptyset$. Here, that $\chi$ has the *finite intersection property* means that, for any finite subset $\chi' \subseteq \chi$, $\bigcap \chi' \neq \emptyset$.

∎

At first, a simple observation is shown to be true.

**Proposition 3.9** Let $\mathcal{F}$ be a general frame. $\mathcal{F}$ is descriptive if and only if $\mathcal{F} \cong \mathfrak{A}_*$ for some modal algebra $\mathfrak{A}$.

**Proof** : Suppose $\mathcal{F}$ is descriptive. Then $\mathcal{F} \cong (\mathcal{F}^*)_*$ holds. Therefore take $\mathcal{F}^*$ for the algebra $\mathfrak{A}$. Conversely suppose $\mathcal{F} \cong \mathfrak{A}_*$. Then $\mathcal{F}^* \cong (\mathfrak{A}_*)^* \cong \mathfrak{A}$, and so, $(\mathcal{F}^*)_* \cong \mathfrak{A}_* \cong \mathcal{F}$. □

Then, the characterization is proved in the following.

**Theorem 3.10** For a general frame $\mathcal{F} = \langle W, R, P \rangle$, $\mathcal{F}$ is descriptive if and only if $\mathcal{F}$ is differentiated, tight, and compact.

**Proof** : Suppose $\mathcal{F}$ is descriptive. By the observation above, there is a modal algebra $\mathfrak{A}$ such that $\mathcal{F} \cong \mathfrak{A}_*$. Therefore, it is enough to be seen that the frame $\mathfrak{A}_* = \langle F_p(A), R_{\mathfrak{A}}, P_{\mathfrak{A}} \rangle$ has these three properties.

For $F, G \in F_p(A)$, it is obvious that $F = G$ implies that $\forall X \in P_{\mathfrak{A}}\big(F \in X$ if and only if $G \in X\big)$. Conversely, suppose $F \not\subseteq G$. Then, there is a point $a \in F$ such that $a \notin G$. Therefore, for $\theta(a) \in P_{\mathfrak{A}}$, $F \in \theta(a)$ but $G \notin \theta(a)$. Hence $\mathfrak{A}_*$ is differentiated.

On the definition of $R_{\mathfrak{A}}$, for any $F, G \in F_p(A)$, ${}_F R_{\mathfrak{A}G}$ if and only if $\forall a \in A, \big(I(a) \in F$ implies $a \in G\big)$, if and only if $\forall \theta(a) \in P_{\mathfrak{A}}\big(F \in \theta(I(a))$ implies $G \in \theta(a)\big)$. Therefore, in order to be seen that $\mathfrak{A}_*$ is tight, it is enough to be proved that $\theta(I(a)) = I_{R_{\mathfrak{A}}}(\theta(a))$. Suppose $F \in \theta(I(a))$, which is equivalent to $I(a) \in F$. Consider any $G \in F_p(A)$ such that ${}_F R_{\mathfrak{A}G}$. Then, by the definition of $R_{\mathfrak{A}}$, $a \in G$ follows, which means that $G \in \theta(a)$. Hence $F \in I_{R_{\mathfrak{A}}}(\theta(a))$ holds. Conversely suppose $F \notin \theta(I(a))$, that is, $I(a) \notin F$. Put $G_0 := \{x \in W \mid I(x) \in F\}$. It is easily checked that $G_0$ is a proper filter and $a \notin G_0$. Therefore by the Lemma 3.3, there exists a prime filter $G$ such that $G_0 \subseteq G$ and $a \notin G$. This $G$ really satisfies that $G \in F_p(A)$, ${}_F R_{\mathfrak{A}G}$ , but $G \notin \theta(a)$. Thus it is the case that $F \notin I_{R_{\mathfrak{A}}}(\theta(a))$. Hence $\theta(I(a)) = I_{R_{\mathfrak{A}}}(\theta(a))$ follows from these facts.

To be shown that $\mathfrak{A}_*$ is compact, take any $\chi \subseteq P_{\mathfrak{A}}$ with the finite intersection property. It can be denoted as $\chi = \{\theta(a) \mid a \in X\}$ for some subset $X \subseteq A$. Let $Z_0 := \{b \in A \mid \bigcap X' \leq b$ for some finite subset $X' \subseteq X\}$. Now, since $\chi$ has the finite intersection property, it can be proved that $Z_0$ is a proper filter in $\mathfrak{A}$, and so, by Lemma 3.3, there exists a prime filter $Z \in F_P(A)$ such that $Z_0 \subseteq Z$. This prime filter $Z$ satisfies that for any $a \in X$, $a \in Z_0 \subseteq Z$, which implies that $Z \in \theta(a)$. Therefore, $Z \in \bigcap_{a \in X} \theta(a) = \bigcap \chi \neq \emptyset$. Thus the frame $\mathfrak{A}_*$ is compact.

On the other hand, suppose $\mathcal{F}$ is differentiated, tight, and compact. The bidual of $\mathcal{F}$ has the following form: $(\mathcal{F}^*)_* = \langle F_p(P), R_{\mathcal{F}^*}, P_{\mathcal{F}^*} \rangle$, where for any $F, G \in F_p(P)$ ${}_F R_{\mathcal{F}^*G}$ if and only if $\forall X \in P$, $\big(I_R(X) \in F$ implies $X \in G\big)$, and $P_{\mathcal{F}^*} = \{\theta(X) \mid X \in P\}$. In the above expression of $P_{\mathcal{F}^*}$, $\theta(X) := \{F \in F_p(A) \mid X \in F\}$ for $X \in P$.

A map $\tau : W \to F_p(P)$ is defined as $\tau(x) := \{X \in P \mid x \in X\}$. It will be shown that this $\tau$ is an isomorphism from $\mathcal{F}$ to $(\mathcal{F}^*)_*$.

The first thing to be checked is that $\tau(x)$ is a prime filter in $P$ for all $x \in W$. Obviously $\emptyset \notin \tau(x)$. For $X, Y \in P$, suppose $X \in \tau(x)$ and $X \subseteq Y$. Then, $x \in X \subseteq Y$, that implies $Y \in \tau(x)$. Moreover, suppose $X, Y \in \tau(x)$, then, $x \in X$ and $x \in Y$ holds, and so, $x \in X \cap Y$. Therefore $X \cap Y \in \tau(x)$. Thus $\tau(x)$ is a proper filter. Suppose $X \cup Y \in \tau(x)$ and $X \notin \tau(x)$. Then $x \in X \cup Y$ but $x \notin X$ follow, which means that $x \in Y$, and so, $Y \in \tau(x)$. Hence $\tau(x)$ is a prime filter in $P$.

**Claim:** $\tau$ is a one to one, onto , and frame-morphism.

Since $\mathcal{F}$ is differentiated, for any $x, y \in W$, $\tau(x) = \tau(y)$ if and only if $\{X \in P \mid x \in X\} = \{X \in P \mid y \in X\}$ if and only if $x = y$, which shows that $\tau$ is one to one.

Take any $F \in F_p(P)$. Since $F$ is proper, $F$ has the intersection property. Because $\mathcal{F}$ is compact, $\bigcap F \neq \emptyset$. Therefore there is a point $x_0 \in W$ such that $x_0 \in \bigcap F = \bigcap_{X \in F} X$. Now, for any $X \in P$, if $X \in F$, then $x_0 \in X$, which means that $X \in \tau(x_0)$. Otherwise, $-X \in F$ because $F$ is a prime filter. Therefore, $x_0 \in -X$, that is, $x_0 \notin X$, and so $X \notin \tau(x_0)$. Thus $F = \tau(x_0)$, which implies that $\tau$ is onto.

For $x, y \in W$, suppose $xRy$. Then since $\mathcal{F}$ is tight, this is equivalent to that for any $X \in P$, $x \in I_R(X)$ implies $y \in X$, but in the term of $\tau$, this can be rewritten to that for any $X \in P$, $I_R(X) \in \tau(x)$ implies $X \in \tau(y)$, which is the definition of ${}_{\tau(x)}R_{\mathcal{F}^*\tau(y)}$. On the other hand, suppose for any $x \in W$, and any $G \in F_p(P)$, ${}_{\tau(x)}R_{\mathcal{F}^*G}$. As a similar argument as just above, there is a point $z \in W$ such that $z \in \bigcap_{X \in G} X$ and $G = \tau(z)$. Then by the fact that ${}_{\tau(x)}R_{\mathcal{F}^*\tau(z)}$, this $z$ satisfies that for any $X \in P$, $I_R(X) \in \tau(x)$ implies $X \in \tau(z)$, which means that $x \in I_R(X)$ implies $z \in X$. Since $\mathcal{F}$ is tight, the last says that $xRz$.

Consider any $Z \in P_{\mathcal{F}^*}$. Since $P_{\mathcal{F}^*} = \{\theta(X) \mid X \in P\}$, there is some $X \in P$ such that $Z = \theta(X)$. Now it has to be proved that $\tau^{-1}(Z) := \{x \in W \mid \tau(x) \in Z\} = X$. For any $x \in W$, $x \in \tau^{-1}(Z)$ if and only if $\tau(x) \in Z$ if and only if $\tau(x) \in \theta(X)$ if and only if $X \in \tau(x)$ if and only if $x \in X$. Thus $\tau^{-1}(Z) = X \in P$. Hence $\tau$ is one to one p-morphism, in order words, an isomorphism. Eventually $(\mathcal{F}^*)_* \cong \mathcal{F}$ is shown to be the case. □

# 4 Algebraic and frame-theoretic conditions for Craig's Interpolation Property and Halldén Completeness

Craig's interpolation property and Halldén completeness are two major syntactical properties of mathematical logics, both of which have been extensively studied by many researchers for a long time ([10], [11], [16], [17], [15], [12], [20]). There already exist a great amount of good results on these topics for modal logics. Here algebraic and frame-theoretic conditions for modal logics to have these properties are discussed; in particular, connection among such conditions are described.

A small notation is introduced before start. For a formula $\varphi \in \Phi$, let $Var(\varphi)$ be the set of all propositional variables that occur in the formula $\varphi$.

A logic $\mathbf{L}$ is *Halldén complete* (H-comp for short), if for any formulas $\varphi$ and $\psi$ such that $Var(\varphi) \cap Var(\psi) = \emptyset$, $\varphi \vee \psi \in \mathbf{L}$ implies either $\varphi \in \mathbf{L}$ or $\psi \in \mathbf{L}$. A logic $\mathbf{L}$ has the *Craig's Interpolation Property* (CIP for short), if for any formulas $\varphi$ and $\psi$, $\varphi \to \psi \in \mathbf{L}$ implies that there exists a formula $\gamma$ ( this is called an *interpolant*) such that both $\varphi \to \gamma \in \mathbf{L}$ and $\gamma \to \psi \in \mathbf{L}$ hold, where $\gamma$ satisfies that $Var(\gamma) \subseteq Var(\varphi) \cap Var(\psi)$.

## 4.1 Algebraic conditions for H-comp and CIP and their equivalence

First, algebraic conditions for H-comp and CIT are introduced and their equivalences for these properties are proved.

**Definition 4.1 (Algebraic Condition for Halldén-complete -ness)** Let $\mathcal{V}$ be a variety of modal algebras. $\mathcal{V}$ has the algebraic condition for Halldén-completeness (ACH for short), if for any algebras $\mathfrak{A}_1, \mathfrak{A}_2$, there exist an algebra $\mathfrak{A} \in \mathcal{V}$ and homomorphisms $f_1 : \mathfrak{A}_1 \to \mathfrak{A}$, and $f_2 : \mathfrak{A}_2 \to \mathfrak{A}$, such that for any $x \in A_1, y \in A_2$, if $x \neq 1$ in $\mathfrak{A}_1$ and $y \neq 1$ in $\mathfrak{A}_2$, then $f_1(x) \cup f_2(y) \neq 1$ in $\mathfrak{A}$. ∎

**Theorem 4.2** Let $\mathcal{V}$ be a non-trivial variety of modal algebras and $\mathbf{L} = \mathbf{L}(\mathcal{V})$. Then the following statements are equivalent.

(1) $\mathcal{V}$ has the (ACH).

(2) $\mathbf{L}$ is H-comp.

**Proof** : (1) $\Rightarrow$ (2): Suppose a formula $\varphi$ is constructed from $p_i$'s and $\psi$ from $q_j$'s. Suppose also $\varphi \notin \mathbf{L}$ and $\psi \notin \mathbf{L}$, and $Var(\varphi) \cap Var(\psi) = \emptyset$. Then there are $\mathfrak{A}_1 \in \mathcal{V}$ and a valuation $v_1$ on $\mathfrak{A}_1$ such that $v_1(\varphi) \neq 1$ in $\mathfrak{A}_1$. Similarly, there are $\mathfrak{A}_2 \in \mathcal{V}$ and a valuation $v_2$ on $\mathfrak{A}_2$ such that $v_2(\psi) \neq 1$ in $\mathfrak{A}_2$. For these two algebras, by (ACH) for $\mathcal{V}$, there are an algebra $\mathfrak{A} \in \mathcal{V}$ and homomorphisms $f_1 : \mathfrak{A}_1 \to \mathfrak{A}$ and $f_2 : \mathfrak{A}_2 \to \mathfrak{A}$. Now since $v_1(\varphi) \neq 1$ and $v_2(\psi) \neq 1$, $f_1(v_1(\varphi)) \cup f_2(v_2(\psi)) \neq 1$ holds. A valuation $v$ on $\mathfrak{A}$ is defined as: $v(p_i) := f_1(v_1(p_i))$ for $p_i$'s and $v(q_j) := f_2(v_2(q_j))$ for $q_j$'s. Then, $v(\varphi \vee \psi) = v(\varphi) \cup v(\psi) = f_1(v_1(\varphi)) \cup f_2(v_2(\psi)) \neq 1$ holds in $\mathfrak{A}$. Hence $\varphi \vee \psi \notin \mathbf{L}$.
(2) $\Rightarrow$ (1): Suppose two modal algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are given. For each element $a \in A_1$, a variable $p_a$ is associated and a language $\mathcal{L}_1$ is determined by $\{p_a \mid a \in A_1\}$. Similarly, for each $b \in A_2$, a variable $q_b$ is associated and a language $\mathcal{L}_2$ is determined by $\{q_b \mid b \in A_2\}$. Let $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$ be the language for the logic $\mathbf{L}$. Consider a valuation $V_1 : \Phi(\mathcal{L}_1) \to \mathfrak{A}_1$ as: $V_1(p_a) := a$, and define $\Sigma_1 := \{\varphi \in \Phi(\mathcal{L}_1) \mid V_1(\varphi) = 1 \text{ in } \mathfrak{A}_1\}$. Similarly, consider a valuation $V_2 : \Phi(\mathcal{L}_2) \to \mathfrak{A}_2$ as: $V_2(q_b) := b$, and define $\Sigma_2 := \{\psi \in \Phi(\mathcal{L}_2) \mid V_2(\psi) = 1 \text{ in } \mathfrak{A}_2\}$.

For $i = 1, 2$, $\Sigma_i$ is closed under modus ponens, because for $\alpha, \beta \in \Phi(\mathcal{L}_i)$, suppose $\alpha, \alpha \to \beta \in \Sigma_i$. Then $V_i(\alpha) = 1$, and $V_i(\alpha \to \beta) = -V_i(\alpha) \cup V_i(\beta) = V_i(\beta) = 1$, and so, $\beta \in \Sigma_i$. Next, $\perp \notin \Sigma_i$ for $i = 1, 2$ because $V_i(\perp) = 0 \neq 1$. This means that both $\Sigma_1$ and $\Sigma_2$ are consistent. Furthermore, $\mathbf{L} \cap \Phi(\mathcal{L}_i) \subseteq \Sigma_i$ for $i = 1, 2$, because, suppose $\alpha \in \mathbf{L} \cap \Phi(\mathcal{L}_i)$. Since $\alpha \in \Phi(\mathcal{L}_i)$, $\alpha$ can be interpretted by $V_i$, and since $\alpha \in \mathbf{L}$, $V_i(\alpha) = 1$, in particular. Hence $\alpha \in \Sigma_i$.

Put $\Sigma := \mathbf{L} \oplus \Sigma_1 \oplus \Sigma_2$. For $i = 1, 2$, for $\varphi \in \Phi(\mathcal{L}_i)$, $\varphi \notin \Sigma_i$ implies $\varphi \notin \Sigma$, because, suppose $\varphi \notin \Sigma_i$ and $\varphi \in \Sigma$. Then, by the deduction theorem, there are $\sigma_1, \cdots, \sigma_m \in \Sigma_i$, and $\tau_1, \cdots, \tau_n \in \Sigma_j$, such that $\left(\Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m\right) \wedge \left(\Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n\right) \to \varphi \in \mathbf{L}$ for some numbers $k_1, k_2, \ldots, k_m$ and $\ell_1, \ell_2, \ldots, \ell_n$, where $\Box^{(k)}\varphi := \varphi \wedge \Box\varphi \wedge \Box^2\varphi \wedge \cdots \wedge \Box^k\varphi$. Put $\mu := \Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m$ and $\nu := \Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n$ for short. Then $\mu \wedge \nu \to \varphi \in \mathbf{L}$ holds, from which $(\mu \to \varphi) \vee \neg\nu \in \mathbf{L}$ can be derived by the classical calculus. Now, since $\mathbf{L}$ is H-comp, either $\mu \to \varphi \in \mathbf{L}$ or $\neg\nu \in \mathbf{L}$. In the former case, because $\mu \in \Sigma_i$ and $\mu \to \varphi \in \mathbf{L} \cap \Phi(\mathcal{L}_i) \subseteq \Sigma_i$, and then since $\Sigma_i$ is closed under modus ponens, $\varphi \in \Sigma_i$ must holds, but this is a contradiction. For the latter case, $\nu \in \Sigma_j$, but $\neg\nu \in \mathbf{L} \cap \Phi(\mathcal{L}_j) \subseteq \Sigma_j$ which lead to a contradiction because $\Sigma_j$ is consistent.

Then, since $\perp \notin \Sigma_1, \Sigma_2$, also $\perp \notin \Sigma$, that means that $\Sigma$ is also consistent.

**Claim:** $\Sigma$ is also H-comp.

Suppose $\varphi \vee \psi \in \Sigma$ for $\varphi \in \Phi(\mathcal{L}_i)$ and $\psi \in \Phi(\mathcal{L}_j)$. Then, similarly by the deduction theorem, there are $\sigma_1, \cdots, \sigma_m \in \Sigma_i$, and $\tau_1, \cdots, \tau_n \in \Sigma_j$, such that $\left(\Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m\right) \wedge \left(\Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n\right) \to \varphi \vee \psi \in \mathbf{L}$ for some numbers $k_1, k_2, \ldots, k_m$ and $\ell_1, \ell_2, \ldots, \ell_n$. Put $\mu := \Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m$ and $\nu := \Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n$ for short.

Then $\mu \wedge \nu \to \varphi \vee \psi \in \mathbf{L}$ holds, from which $(\mu \to \varphi) \vee (\nu \to \psi) \in \mathbf{L}$ can be derived by the classical calculus. Since $\mathbf{L}$ is H-comp, either $\mu \to \varphi \in \mathbf{L}$ or $\nu \to \psi \in \mathbf{L}$ holds. In the former case, because $\mu \in \Sigma_i$ and $\mu \to \varphi \in \mathbf{L} \cap \Phi(\mathcal{L}_i) \subseteq \Sigma_i$, $\varphi \in \Sigma_i \subseteq \Sigma$ can be deduced. Similarly in the latter case, $\psi \in \Sigma_j \subseteq \Sigma$ is deducible. Hence it is shown that $\Sigma$ is H-comp.

Define a modal algebra $\mathfrak{A} := \langle A, \cap, \cup, -, I, 0, 1 \rangle$ as follows: $A := \{||\alpha|| \mid \alpha \in \Phi(\mathcal{L})\}$, where $||\alpha|| := \{\beta \in \Phi(\mathcal{L}) \mid \alpha \leftrightarrow \beta \in \Sigma\}$. For operators, $1 := ||\top||$, $-||\alpha|| := ||\neg\alpha||$, $||\alpha|| \cap ||\beta|| := ||\alpha \wedge \beta||$, and $I(||\alpha||) := ||\Box\alpha||$. These operations are easily shown to be well defined, and $\mathfrak{A} \in \mathcal{V}$, because, $\mathbf{L} \subseteq \Sigma$, $\mathfrak{A} \models \mathbf{L}$.

Furthermore, maps $f_1 : \mathfrak{A}_1 \to \mathfrak{A}$ and $f_2 : \mathfrak{A}_2 \to \mathfrak{A}$ are defined as: $f_1(a) := ||p_a||$ for $a \in A_1$, $f_2(b) := ||q_b||$ for $b \in A_2$. Then a simple calculation shows that these are indeed homomorphisms.

Now take any $x(\neq 1) \in \mathfrak{A}_1$ and any $y(\neq 1) \in \mathfrak{A}_2$. Denote $||\varphi|| := f_1(x) = ||p_x||$ and $||\psi|| := f_2(y) = ||q_y||$. Because $x \neq 1$ in $\mathfrak{A}_1$, $\varphi \notin \Sigma_1$, and so, $\varphi \notin \Sigma$. Similarly $\psi \notin \Sigma$ is deduced. Here, if $||\varphi \vee \psi|| = 1$, then $\varphi \vee \psi \in \Sigma$ must be the case. However, since $\Sigma$ is H-comp, either $\varphi \in \Sigma$ or $\psi \in \Sigma$ must hold. This is a contradiction. Therefore $f_1(x) \cup f_2(y) = ||\varphi \vee \psi|| \neq 1$. □

**Definition 4.3 (Amalgamation Property and Super Amalgamation Property)** Let $\mathcal{V}$ be a variety of modal algebras.

(1) $\mathcal{V}$ has the Amalgamation Property (AP for short), if for every algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$ such that there are embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1, f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$, then there exist an algebra $\mathfrak{A} \in \mathcal{V}$, and embeddings $g_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $g_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that $(g_1 \circ f_1)(x) = (g_2 \circ f_2)(x)$ for any $x \in A_0$.

(2) $\mathcal{V}$ has the Super Amalgamation Property (SAP for short), $\mathcal{V}$ has the amalgamation property, and also satisfies: for any $x \in A_i$ and any $y \in A_j$ ($\{i,j\} = \{1,2\}$), $g_i(x) \leq g_j(y)$ in $\mathfrak{A}$ implies that there exists $z \in A_0$ such that $x \leq_i f_i(z)$ in $\mathfrak{A}_i$ and $f_j(z) \leq_j y$ in $\mathfrak{A}_j$. ∎

**Theorem 4.4** Let $\mathcal{V}$ be a non-trivial variety of modal algebras and $\mathbf{L} = \mathbf{L}(\mathcal{V})$. Then the following statements are equivalent.

(1) $\mathcal{V}$ has the (SAP).

(2) $\mathbf{L}$ has the CIP.

**Proof** : (1) $\Rightarrow$ (2): Consider formulas $\varphi = \varphi(p_1, \cdots, p_\ell, r_1, \cdots, r_n)$ constructed only from $\{p_1, \cdots, p_\ell, r_1, \cdots, r_n\}$ and $\psi = \psi(q_1, \cdots, q_m, r_1, \cdots, r_n)$ constructed only from $\{q_1, \cdots, q_m, r_1, \cdots, r_n\}$. Suppose there exists no $\chi = \chi(r_1, \cdots, r_n)$ such that both $\varphi \to \chi \in \mathbf{L}$ and $\chi \to \psi \in \mathbf{L}$ hold. Then it is enough to be seen that $\varphi \to \psi \notin \mathbf{L}$. Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ be free-$\mathcal{V}$ algebras generated by the sets $\{c_1, \cdots, c_n\}$, $\{a_1, \cdots, a_\ell, c_1, \cdots, c_n\}$, and $\{b_1, \cdots, b_m, c_1, \cdots, c_n\}$ respectively. Then clearly $\mathfrak{A}_0$ is embedded into both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by identity maps. So by the (SAP) of $\mathcal{V}$, there exist an algebra $\mathfrak{A} \in \mathcal{V}$ and embeddings $g_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $g_2 : \mathfrak{A}_2 \to \mathfrak{A}$ with some properties.
**Claim:** $g_s(\hat{\varphi}(a_1, \cdots, a_\ell, c_1, \cdots, c_n)) \not\leq g_t(\hat{\psi}(b_1, \cdots, b_m, c_1, \cdots, c_n))$ for $\{s,t\} = \{1,2\}$, where $\hat{\varphi}$ ($\hat{\psi}$) is a term in the free-$\mathcal{V}$ algebra corresponding to $\varphi$ ($\psi$).

Suppose this inequality holds. Then, by (SAP) of $\mathcal{V}$, there exists $d \in A_0$ such that $g_s(\hat{\varphi}(a_1, \cdots,$
$a_\ell, c_1, \cdots, c_n)) \leq d$ in $\mathfrak{A}_s$ and $d \leq g_t(\hat{\psi}(b_1, \cdots, b_m, c_1, \cdots, c_n))$ in $\mathfrak{A}_t$. In terms of logic, free-$\mathcal{V}$ algebra corresponds to the Lindenbaum $\mathbf{L}$-algebra, $\leq$ to the deducibility, and $d$ to a formula $\chi$ which is constructed from $\{c_1, \cdots, c_n\}$. Therefore,that the inequalities hold means

$\varphi \to \chi \in \mathbf{L}$ and $\chi \to \psi \in \mathbf{L}$ hold. This is a contradiction.

Now, define a valuation $v$ on $\mathfrak{A}$ as: $v(p_i) := g_s(a_i)$ for $i = 1, \cdots, a_\ell$, $v(q_j) := g_t(b_j)$ for $j = 1, \cdots, m$, and $v(r_k) := g_s(c_k) = g_t(c_k)$ for $k = 1, \cdots, n$, where $\{s, t\} = \{1, 2\}$. Since $g_1, g_2$ are homomorphisms, $v(\varphi) \not\leq v(\psi)$ in $\mathfrak{A}$, which implies that $\varphi \to \psi \notin \mathbf{L}$.

(2) $\Rightarrow$ (1): Suppose for $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$, there are embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. Here, both $f_1$ and $f_2$ may be assumed to be identity maps, that is, $f_1(x) = f_2(x) = x$ for all $x \in A_0$. For $i = 0, 1, 2$, a variable $p^i_a$ is associated with each element $a \in A_i$ in such a way that for $a \in A_0$, $p^1_a = p^2_a = p^0_a$. Denote the language with the variables $p^i_a$ for $a \in A_i$ by $\mathcal{L}_i$ $(i = 0, 1, 2)$ and $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$. Terms and formulas are not distinguished here, and $\mathcal{L}$ is assumed to be the language of the logic $\mathbf{L}$.

A valuation $V_i$ of $\mathcal{L}_i$ on $\mathfrak{A}_i$ is defined as: $V_i(p^i_a) := a$ and put $\Sigma_i := \{\varphi \in \Phi(\mathcal{L}_i) \mid V_i(\varphi) = 1\}$ for $i = 1, 2$. Then as in the proof of Theorem 4.2, it can be shown that $\mathbf{L} \cap \Phi(\mathcal{L}_i) \subseteq \Sigma_i$ and that $\Sigma_i$ is closed under modus ponens. Let $\Sigma := \mathbf{L} \oplus \Sigma_1 \oplus \Sigma_2$.

**Claim:** For $\varphi \in \Phi(\mathcal{L}i)$ and $\psi \in \Phi(\mathcal{L}_j)$ $(\{i, j\} = \{1, 2\})$, $\varphi \to \psi \in \Sigma$ if and only if $\exists \chi \in \Phi(\mathcal{L}_0)\big(\varphi \to \chi \in \Sigma_i$ and $\chi \to \psi \in \Sigma_j\big)$.

If part is trivial, because $\Sigma_i, \Sigma_j \subseteq \Sigma$, which is closed under modus ponens.

For only if part, suppose $\varphi \to \psi \in \Sigma$. Then, by the deduction theorem, there are $\sigma_1, \cdots, \sigma_m \in \Sigma_i$, and $\tau_1, \cdots, \tau_n \in \Sigma_j$, such that $\big(\Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m\big) \wedge \big(\Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n\big) \wedge \varphi \to \psi \in \mathbf{L}$ for some numbers $k_1, k_2, \ldots, k_m$ and $\ell_1, \ell_2, \ldots, \ell_n$. Put $\mu := \Box^{(k_1)}\sigma_1 \wedge \cdots \wedge \Box^{(k_m)}\sigma_m$ and $\nu := \Box^{(\ell_1)}\tau_1 \wedge \cdots \wedge \Box^{(\ell_n)}\tau_n$ for short. Then $(\mu \wedge \nu \wedge \varphi) \to \psi \in \mathbf{L}$ holds, from which $(\mu \wedge \varphi) \to (\nu \to \psi) \in \mathbf{L}$ can be derived by the classical calculus. Since $\mathbf{L}$ has the CIP, there exists a formula

$\chi \in \Phi(\mathcal{L}_0)$ such that $\mu \wedge \varphi \to \chi \in \mathbf{L}$ and $\chi \to (\nu \to \psi) \in \mathbf{L}$. By the former, $\mu \to (\varphi \to \chi) \in \Sigma_i$, from which $\varphi \to \chi \in \Sigma_i$ is deduced since $\mu \in \Sigma_i$. Similarly, by the latter $\nu \to (\chi \to \psi) \in \Sigma_j$ is deduced, from which $\chi \to \psi \in \Sigma_j$ is deduced since $\nu \in \Sigma_j$.

Note here that if $\varphi = \top$ in particular, $\chi = \top$ holds. In this case, this claim means that $\varphi \in \Sigma$ implies $\psi \in \Sigma_j$. Thus $\Sigma \cap \Phi(\mathcal{L}_j) = \Sigma_j$ for $j = 1, 2$.

Construct an algebra $\mathfrak{A} = \langle A, \cap, \cup, -, I, 0, 1\rangle$ as follows: $A := \{||\varphi|| \mid \varphi \in \Phi(\mathcal{L})\}$, where $||\varphi|| := \{\psi \in \Phi(\mathcal{L}) \mid \varphi \leftrightarrow \psi \in \Sigma\}$. For operators, $1 := ||\top||$, $-||\varphi|| := ||\neg\varphi||$, $||\varphi|| \cap ||\psi|| := ||\varphi \wedge \psi||$, and $I(||\varphi||) := ||\Box\varphi||$. This $\mathfrak{A}$ is well defined, and $\mathfrak{A} \in \mathcal{V}$ because $\mathbf{L} \subseteq \Sigma$ and $\mathfrak{A} \models \mathbf{L}$. Furthermore, define maps $g_i : \mathfrak{A}_i \to \mathfrak{A}$ by $g_i(a) := ||p^i_a||$ for $i = 1, 2$.

Then this $g_i$ is one to one, because for $a, b \in A_i$, suppose $g_i(a) = g_i(b)$. Then $||p^i_a|| = ||p^i_b||$, which means that $p^i_a \leftrightarrow p^i_b \in \Sigma \cap \Phi(\mathcal{L}_i) = \Sigma_i$, and so, $V_i(p^i_a \leftrightarrow p^i_b) = 1$. Therefore $a = V_i(p^i_a) = V_i(p^i_b) = b$. Moreover, simple calculation shows that $g_i$ is a homomorphism. Eventually $g_i$ turns out to be an embedding. For $a \in A_0$, $g_1(f_1(a)) = g_1(a) = ||p^1_a|| = ||p^0_a|| = ||p^2_a|| = g_2(a) = g_2(f_2(a))$ holds.

Suppose for $a \in A_i$ and $b \in A_j$ ($\{i, j\} = \{1, 2\}$), $g_i(a) \leq g_j(b)$ in $\mathfrak{A}$. This inequality means $||p^i_a|| \leq ||p^j_b||$, which implies that $p^i_a \to p^j_b \in \Sigma$. Then, there exists a formula $\chi \in \Phi(\mathcal{L}_0)$ such that $p^i_a \to \chi \in \Sigma_i$ and $\chi \to p^j_b \in \Sigma_j$. Put $c := V_i(\chi) = V_j(\chi) \in A_0$. Then $V_i(p^i_a) \leq V_i(\chi)$ in $\mathfrak{A}_i$ and $V_j(\chi) \leq V_j(p^j_b)$ in $\mathfrak{A}_j$. Thus $a \leq_i f_i(c)$ and $f_j(c) \leq_j b$. $\Box$

## 4.2 Frame-theoretic conditions for H-comp and CIP and their equivalence to algebraic ones

The algebraic conditions for H-comp and CIP can be rewritten in terms of general frames. Each frame-theoretic condition is proved to be equivalent to the corresponding algebraic one. First the frame-theoretic condition for H-comp is introduced and it is shown that this is equivalent to the (ACH).

**Definition 4.5 (Frame-theoretic Condition for H-comp)** Let $\mathcal{K}$ be a class of general frames. $\mathcal{K}$ has the frame-theoretic condition for H-comp (FCH for short), if for any frames $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$ and for any points $x \in W_1$ and $y \in W_2$, there exist a frame $\mathcal{F} \in \mathcal{K}$ and frame-morphisms $\nu_1 : \mathcal{F} \to \mathcal{F}_1$ and $\nu_2 : \mathcal{F} \to \mathcal{F}_2$, and a point $z \in W$, such that $\nu_1(z) = x$ and $\nu_2(z) = y$. ∎

This condition is also expressed as "The class $\mathcal{K}$ of frames is *closed under p-morphic fusion*" ([12]). But for H-comp, the frame-morphisms might not be one to one. Hence the maps appearing in the above definition are not p-morphisms.

In order to be shown that (FCH) is equivalent to (ACH), the notion of direct product of algebras is needed. For a class $\mathcal{C} := \{\mathfrak{A}_i\}_{i\in I}$ of modal algebras, the *direct product* $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$ of all members of $\mathcal{C}$ is defined as follows: its underlying set is the direct product $\prod_{i\in I} A_i$, where $A_i$ is the underlying set of the algebra $\mathfrak{A}_i$ for each $i \in I$. For every element $a \in \prod_{i\in I} A_i$, $a(i) := a_i \in A_i$ is called an *i-th coordinate* of $a$. Each operation is defined coordinate-wise: for each $i \in I$, $1(i) := 1_i \in A_i$, for $a, b \in \prod_{i\in I} A_i$, $(a \wedge b)(i) := a(i) \wedge_i b(i)$, $(-a)(i) := -_i a(i)$, and

$(I(a))(i) := I_i(a(i))$. For a subclass $\mathcal{C} := \{\mathfrak{A}_i\}_{i \in I}$ of a variety $\mathcal{V}$, the direct product $\prod_{i \in I} \mathfrak{A}_i$ is again a member of $\mathcal{V}$.

For a class $\mathcal{C}$ of modal algebras, denote $\mathcal{C}_* := \{\mathfrak{A}_* \mid \mathfrak{A} \in \mathcal{C}\}$, and similarly for a class $\mathcal{K}$ of general frames, denote $\mathcal{K}^* := \{\mathcal{F}^* \mid \mathcal{F} \in \mathcal{K}\}$.

**Theorem 4.6**

(1) For a class $\mathcal{K}$ of descriptive general frames, if $\mathcal{K}^*$ has the (ACH), then $\mathcal{K}$ has the (FCH).

(2) For a class $\mathcal{C}$ of modal algebras, if $\mathcal{C}_*$ has the (FCH), then $\mathcal{C}$ has the (ACH).

**Proof** : (1): Suppose there are frames $\mathcal{F}_1 := \langle W_1, R_1, P_1 \rangle$ and $\mathcal{F}_2 := \langle W_2, R_2, P_2 \rangle$ in $\mathcal{K}$, and points $x \in W_1, y \in W_2$. Because $\mathcal{F}_1{}^*, \mathcal{F}_2{}^* \in \mathcal{K}^*$, and so, by (ACH) of $\mathcal{K}^*$, there are a frame $\mathcal{F} \in \mathcal{K}$ ($\mathcal{F} := \langle W, R, P \rangle$) and homomorphisms $f_1 : \mathcal{F}_1{}^* \to \mathcal{F}^*$, $f_2 : \mathcal{F}_2{}^* \to \mathcal{F}^*$. By Lemma 3.6, maps $\nu_1{}'(F) := \{X \in P_1 | f_1(X) \in F\}$ and $\nu_2{}'(G) := \{Y \in P_2 | f_2(Y) \in G\}$ for $F, G \in F_p(P)$ are frame-morphisms. But since $\mathcal{F}$ is descriptive, using the isomorphism $\tau : \mathcal{F} \to (\mathcal{F}^*)_*$ appeared in the proof of Theorem 3.10, that is, $\tau_1 : \mathcal{F}_1 \to (\mathcal{F}_{1*})^*$, $\tau_2 : \mathcal{F}_2 \to (\mathcal{F}_{2*})^*$, $\nu_1{}'$ and $\nu_2{}'$ should be modified as: $\nu_1(u) := \tau_1{}^{-1}(\{X \in P_1 \mid f_1(X) \in \tau_1(u)\})$, and $\nu_2(v) := \tau_2{}^{-1}(\{Y \in P_2 \mid f_2(X) \in \tau_2(v)\})$ for $u, v \in W$. Put $\mathcal{S} := \{f_1(X) \mid X \in P_1, x \in X\} \cup \{f_2(Y) \mid Y \in P_2, y \in Y\}$.

**Claim:** $\mathcal{S}$ has the finite intersection property.

Consider any finite members of $\mathcal{S}$, that is, $X_1, \ldots, X_n \in P_1$ such that $x \in X_i$ for all $1 \leq i \leq n$ and $Y_1, \ldots, Y_m \in P_2$ such that $y \in Y_j$ for all $1 \leq j \leq m$. Then, $\bigcap_{i=1}^{n} f_1(X_i) \cap \bigcap_{j=1}^{m} f_2(Y_j) =$
$f_1(X_1 \cap \cdots \cap X_n) \cap f_2(Y_1 \cap \cdots \cap Y_m) = f_1(X) \cap f_2(Y) \neq \emptyset$, (where $X := X_1 \cap \cdots \cap X_n$ and $Y := Y_1 \cap \cdots \cap Y_m$) because, since $x \in X$ and

$y \in Y$, $-X(\neq W_1) \in P_1$ and $-Y(\neq W_2) \in P_2$, and so, by (ACH) of $\mathcal{K}^*$, $f_1(-X) \cup f_2(-Y) \neq W$, which implies that $f_1(X) \cap f_2(Y) \neq \emptyset$. Hence $\mathcal{S}$ has the finite intersection property.

Now, because $\mathcal{F}$ is compact, there exists a point $z \in W$ such that $z \in \bigcap\{f_1(X) \mid X \in P_1, x \in X\}$ $\cap \bigcap\{f_2(Y) \mid Y \in P_2, y \in Y\}$. By the duality between $f_k$ and $\nu_k$, it is easily seen that $\nu_k(w) \in Z$ if and only if $w \in f_k(Z)$ for $k = 1, 2$. Therefore, this point $z$ has the following property: for any $X \in P_1\big(x \in X$ implies $\nu_1(z) \in X\big)$, and for any $Y \in P_2\big(y \in Y$ implies $\nu_2(z) \in Y\big)$. Here, suppose $x \neq \nu_1(z)$. Then since $\mathcal{F}_1$ is differentiated, there is $T \in P_1$ such that $x \in T$ and $\nu_1(z) \notin T$, that leads to a contradiction to the above property of $z$. Thus $x = \nu_1(z)$. By the same reasoning $y = \nu_2(z)$ can be also deduced.

(2): Suppose there are algebras $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$. Let $\Lambda := \{\langle x, y\rangle \mid x \in A_1, x \neq 1, y \in A_2, y \neq 1\}$. Then for each $\lambda := \langle x_\lambda, y_\lambda\rangle \in \Lambda$, since $-x_\lambda \neq 0$, there exists a prime filter $F_\lambda$ in $\mathfrak{A}_1$ such that $-x_\lambda \in F_\lambda$. Similarly, since $-y_\lambda \neq 0$, there exists a prime filter $G_\lambda$ in $\mathfrak{A}_2$ such that $-y_\lambda \in G_\lambda$. Of course, $F_\lambda$ is a point in the frame $\mathfrak{A}_{1*}$, and $G_\lambda$ is a point in the frame $\mathfrak{A}_{2*}$. By the (FCH) of $\mathcal{C}_*$, there exist an algebra $\mathfrak{A}^\lambda \in \mathcal{C}$ and frame-morphisms ${\nu_1}^\lambda : \mathfrak{A}_{1*} \to {\mathfrak{A}^\lambda}_*$ and ${\nu_2}^\lambda : \mathfrak{A}_{2*} \to {\mathfrak{A}^\lambda}_*$, and a prime filter $H^\lambda$ in $\mathfrak{A}^\lambda$ such that ${\nu_1}^\lambda(H_\lambda) = F_\lambda$ and ${\nu_2}^\lambda(H_\lambda) = G_\lambda$ hold. Then by Lemma 3.7, maps $({f_1}^\lambda)' : {\mathfrak{A}_{1*}}^* \to ({\mathfrak{A}^\lambda}_*)^*$ and $({f_2}^\lambda)' : {\mathfrak{A}_{2*}}^* \to ({\mathfrak{A}^\lambda}_*)^*$ defined below are homomorphisms, that is, $({f_1}^\lambda)'(X) := \{E \in F_p(A) \mid {\nu_1}^\lambda(E) \in X\}$ for $X \in P_{\mathfrak{A}_1}$, $({f_2}^\lambda)'(X) := \{E \in F_p(A) | {\nu_2}^\lambda(E) \in X\}$ for $X \in P_{\mathfrak{A}_2}$. By using the isomorphism $\theta^\lambda : \mathfrak{A}^\lambda \to ({\mathfrak{A}^\lambda}_*)^*$ in Proposition 3.4, $({f_1}^\lambda)', ({f_2}^\lambda)'$ should be modified as: ${f_1}^\lambda(a) := {\theta^\lambda}^{-1}\big(\{E \in F_p(A) \mid a \in {\nu_1}^\lambda(E)\}\big)$ and ${f_2}^\lambda(b) := {\theta^\lambda}^{-1}\big(\{E \in F_p(A) \mid a \in {\nu_2}^\lambda(E)\}\big)$. Consider the direct product $\mathfrak{A} := \prod_{\lambda\in\Lambda} \mathfrak{A}^\lambda$, and

$f_1 : \mathfrak{A}_1 \to \mathfrak{A}$, whose $\lambda$-th component is $(f_1)(\lambda) = {f_1}^{\lambda}$, $f_2 : \mathfrak{A}_2 \to \mathfrak{A}$,whose $\lambda$-th component is $(f_2)(\lambda) = {f_2}^{\lambda}$. It is easily checked that these are homomorphisms. Here, take any $x(\neq 1)$ in $A_1$ and any $y(\neq 1)$ in $A_2$, and put $z := f_1(x) \cup f_2(y) \in \mathfrak{A}$. Then there is $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} = x$ and $y_{\lambda_0} = y$. Now $z_{\lambda_0} = {f_1}^{\lambda_0}(x_{\lambda_0}) \cup {f_2}^{\lambda_0}(y_{\lambda_0})$. Since $x_{\lambda_0} \notin F_{\lambda_0} = {\nu_1}^{\lambda_0}(H_{\lambda_0})$ and $y_{\lambda_0} \notin G_{\lambda_0} = {\nu_2}^{\lambda_0}(H_{\lambda_0})$, $H_{\lambda_0} \notin \theta^{\lambda_0}(z_{\lambda_0}) = \{E \in F_p(A) \,|\, x_{\lambda_0} \in {\nu_1}^{\lambda_0}(E)\} \cup \{E \in F_p(A) \,|\, y_{\lambda_0} \in {\nu_2}^{\lambda_0}(E)\}$. Therefore $z_{\lambda_0} \neq 1$ in $\mathfrak{A}_{\lambda_0}$. Hence $z = f_1(x) \cup f_2(y) \neq 1$ in $\mathfrak{A}$. □

On the other hand, the frame-theoretic condition for CIP is introduced in the following:

**Definition 4.7 (Frame-theoretic Condition for CIP)** Let $\mathcal{K}$ be a class of general frames.

(1) $\mathcal{K}$ has the Amalgamation Property for Frames (APF for short), if for any $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$, and for any p-morphisms $\theta_1 : \mathcal{F}_1 \to \mathcal{F}_0$, $\theta_2 : \mathcal{F}_2 \to \mathcal{F}_0$, there exist $\mathcal{F} \in \mathcal{K}$ and p-morphisms $\tau_1 : \mathcal{F} \to \mathcal{F}_1$, $\tau_2 : \mathcal{F} \to \mathcal{F}_2$ such that $\theta_1 \circ \tau_1 = \theta_2 \circ \tau_2$.

(2) $\mathcal{K}$ has the Super Amalgamation Property for Frames (SAPF for short), if $\mathcal{K}$ has the (APF) and also satisfies that for any $x \in W_1$ and for any $y \in W_2$, $\theta_1(x) = \theta_2(y)$ implies that $\tau_1(z) = x$ in $\mathcal{F}_1$ and $\tau_2(z) = y$ in $\mathcal{F}_2$ for some $z \in W$.

∎

The equivalence of frame condition for CIT is shown by two steps: the first step is to be proved the equivalence between (AP) and (APF), the second, (SAP) and (SAPF).

**Theorem 4.8**

(1) For a class $\mathcal{K}$ of descriptive general frames, if $\mathcal{K}_*$ has the (AP), then $\mathcal{K}$ has the (APF).

(2) For a class $\mathcal{C}$ of modal algebras, if $\mathcal{C}_*$ has the (APF), then $\mathcal{C}$ has the (AP).

**Proof** : (1): Suppose for frames $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$, there are p-morphisms $\mu_1 : \mathcal{F}_1 \to \mathcal{F}_0$ and $\mu_2 : \mathcal{F}_2 \to \mathcal{F}_0$. Then By Lemma 3.7, $g_1(X) := \{a \in W_1 \mid \mu_1(a) \in X\}$ and $g_2(Y) := \{b \in W_2 \mid \mu_2(b) \in Y\}$ for $X, Y \in W_0$ are embeddings. ($g_1 : \mathcal{F}_0{}^* \to \mathcal{F}_1{}^*$, $g_2 : \mathcal{F}_0{}^* \to \mathcal{F}_2{}^*$) By (AP) of $\mathcal{K}^*$, there are an algebra $\mathcal{F}^* \in \mathcal{K}^*$ and embeddings $f_1 : \mathcal{F}_1{}^* \to \mathcal{F}^*, f_2 : \mathcal{F}_2{}^* \to \mathcal{F}^*$ such that $f_1 \circ g_1 = f_2 \circ g_2$. Here by Lemma 3.6, $\nu_1{}'(F) := \{X \in P_1 \mid f_1(X) \in F\}$ and $\nu_2{}'(G) := \{Y \in P_2 \mid f_2(Y) \in G\}$ for $F, G \in F_p(P)$ are p-morphisms. ($\nu_1{}' : (\mathcal{F}_*)^* \to (\mathcal{F}_{1*})^*, \nu_2{}' : (\mathcal{F}_*)^* \to (\mathcal{F}_{2*})^*$) Since $\mathcal{F}$ is descriptive, by using an isomorphism $\tau_k : \mathcal{F}_k \to (\mathcal{F}_k{}^*)_*$ defined in the proof of Theorem 3.10 as $\tau_k(x) := \{X \in P_k \mid x \in X\}$ for $k = 1, 2$, $\nu_1{}'$ and $\nu_2{}'$ should be modified as $\nu_1(x) := \tau_1{}^{-1}(\{X \in P_1 | f_1(X) \in \tau_1(x)\})$, $\nu_2(y) := \tau_2{}^{-1}(\{Y \in P_2 \mid f_2(Y) \in \tau_2(y)\})$.
**Claim:** for any $a \in W$ and for any $X \in P$ in $\mathcal{F}$, $X \in \tau \circ \mu_1 \circ \nu_1(a)$ implies $X \in \tau \circ \mu_2 \circ \nu_2(a)$, where $\tau : \mathcal{F} \to (\mathcal{F}^*)_*$ defined as $\tau(x) := \{X \in P \mid x \in X\}$.

Because, $X \in \tau \circ \mu_1 \circ \nu_1(a)$ if and only if $\mu_1 \circ \nu_1(a) \in X$ if and only if $\nu_1(a) \in g_1(X)$ if and only if $\tau_1{}^{-1}(\{X \in P_1 \mid f_1(X) \in \tau_1(a)\}) \in g_1(X)$. Here, put $b := \tau_1{}^{-1}(\{X \in P_1 \mid f_1(X) \in \tau_1(a)\})$. Then, $b \in g_1(X)$ and $\tau_1(b) = \{Z \in P_1 \mid f_1(Z) \in \tau_1(a)\}$. This means that for any $Z \in P_1$, $b \in Z$ if and only if $a \in f_1(Z)$. Since $b \in g_1(X)$, if $g_1(X)$ is taken for $Z$, $a \in f_1 \circ g_1(X) = f_2 \circ g_2(X)$. Put $Y_0 := g_2(X)$. Then $a \in f_2(Y_0)$, which is equivalent to $f_2(Y_0) \in \tau_2(a)$. Again, put $c = \nu_2(a) = \tau_2{}^{-1}(\{Y \in$

$P_2 | f_2(Y) \in \tau_2(a)\}$). Then $\tau_2(c) = \{Y \in P_2 | f_2(Y) \in \tau_2(a)\}$. Therefore, $Y_0 \in \tau_2(c)$, and so, $c = \nu_2(a) \in Y_0 = g_2(X)$. Eventually $\mu_2 \circ \nu_2(a) \in X$, that implies $X \in \tau \circ \mu_2 \circ \nu_2(a)$.

Thus it is the case that $\tau \circ \mu_1 \circ \nu_1 = \tau \circ \mu_2 \circ \nu_2$. Since $\tau$ is one to one, it is shown that $\mu_1 \circ \nu_1 = \mu_2 \circ \nu_2$.

(2): Suppose there are algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$ and embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$, $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. By Lemma 3.6, maps $g_1 : \mathfrak{A}_{1*} \to \mathfrak{A}_{0*}$ defined by $g_1(F) := \{a \in A_0 \mid f_1(a) \in F\}$ for $F \in F_p(A_1)$ and $g_2 : \mathfrak{A}_{2*} \to \mathfrak{A}_{0*}$ defined by $g_2(G) := \{b \in A_0 \mid f_2(b) \in G\}$ for $G \in F_p(A_2)$ are p-morphisms. By (APF) of $\mathcal{C}_*$, there exist $\mathfrak{A}_* \in \mathcal{C}_*$ and p-morphisms $\sigma_1 : \mathfrak{A}_* \to \mathfrak{A}_{1*}$ and $\sigma_2 : \mathfrak{A}_* \to \mathfrak{A}_{2*}$ such that $g_1 \circ \sigma_1 = g_2 \circ \sigma_2$. Then, by Lemma 3.7, $h_k{}'(X) := \{F \in F_p(A) \mid \sigma_k(F) \in X\}$ for $F \in P_{\mathfrak{A}_k}$ is an embeddings from $(\mathfrak{A}_{k*})^*$ into $(\mathfrak{A}_*)^*$ $(k = 1, 2)$. By using the isomorphism $\theta : \mathfrak{A} \to (\mathfrak{A}_*)^*$ defined as $\theta(a) := \{F \in F_p(A) \mid a \in F\}$ for $a \in A$ in Proposition 3.4, $h_k{}'$ should be modified as: $h_k(a) := \theta^{-1}(\{F \in F_p(A) \mid \sigma_k(F) \in \theta_k(a)\})$ $(h_k : \mathfrak{A}_k \to \mathfrak{A})$ for $k = 1, 2$, where $\theta_k : \mathfrak{A}_k \to (\mathfrak{A}_{k*})^*$ is the isomorphism for $\mathfrak{A}_k$.

Then, for any $F \in F_p(A)$ and any $a \in A$, $F \in \theta \circ h_1 \circ f_1(a)$ if and only if $\sigma_1(F) \in \theta(f_1(a))$ if and only if $f_1(a) \in \sigma_1(F)$ if and only if $a \in g_1 \circ \sigma_1(F) = g_2 \circ \sigma_2(F)$ if and only if $F \in \theta \circ h_2 \circ f_2(a)$, which implies that $\theta \circ h_1 \circ f_1 = \theta \circ h_2 \circ f_2$. Here, since $\theta$ is one to one, it can be proved that $h_1 \circ f_1 = h_2 \circ f_2$. □

Before the equivalence of (SAP) and (SAPF) is proved, an extremely important lemma related to CIP is shown to be valid here.

**Lemma 4.9** Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ be modal algebras, where $\mathfrak{A}_0$ is a subalgebra of both $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Suppose for $a \in A_1$ and $b \in A_2$, there is no $c \in A_0$ such that both $a \leq_1 c$ and $c \leq_2 b$ hold. Then there exist prime filters $F_1$ in $\mathfrak{A}_1$ and $F_2$ in $\mathfrak{A}_2$ such that $a \in F_1, b \notin F_2$ and $F_1 \cap A_0 = F_2 \cap A_0$.

**Proof** : Put $X := \{x \in A_0 \mid a \leq_1 x\}$, and $Y := \{y \in A_0 \mid y \leq_2 b\}$. Then by the assumption, $X \cap Y = \emptyset$. Consider a family $\mathfrak{J}_2 := \{J \subseteq A_2 \mid J = \downarrow_2(J), \{b\} \cup Y \subseteq J, X \cap J = \emptyset\}$, where $\downarrow_2(Z) := \{u \in A_2 \mid u \leq_2 \bigcup Z_0$ for some finite subset $Z_0 \subseteq Z\}$.

Now, $\downarrow_2(\{b\}) = \downarrow_2(\downarrow_2(\{b\}))$, $\{b\} \cup Y \subseteq \downarrow_2(\{b\})$, $X \cap \downarrow_2(\{b\}) = \emptyset$. Thus $\downarrow_2(\{b\}) \in \mathfrak{J}_2$, which means that $\mathfrak{J}_2 \neq \emptyset$. Take a chain $\mathcal{C} := \{Z_i\}_{i \in \omega}$ of elements in $\mathfrak{J}_2$, and put $Z := \bigcup \mathcal{C}$.

**Claim 1:** $Z$ is a maximal element in the chain $\mathcal{C}$.

Because, for any $u \in \downarrow_2(Z)$, there exist $z_1, \ldots, z_n \in \bigcup \mathcal{C}$, such that $u \leq_2 z_1 \cup \cdots \cup z_n$ holds. Since $\mathcal{C}$ is a chain, for some number $j \in \omega$ $z_1, \cdots, z_n \in Z_j$. Therefore $u \in \downarrow_2(Z_j) = Z_j \subseteq Z$. Thus $Z = \downarrow_2(Z)$. It is obvious that $\{b\} \cup Y \subseteq Z$. Suppose $Z \cap X \neq \emptyset$. Then there exists $x \in X$ such that $x \in Z$. Then $x \in Z_k$ for some $k \in \omega$ but this implies that $X \cap Z_k \neq \emptyset$. This is a contradiction. Hence $Z \in \mathfrak{J}_2$. Clearly $Z$ is maximal in $\mathcal{C}$.

By Zorn's lemma, there is a maximal element $J_2 \in \mathfrak{J}_2$.

**Claim 2:** $J_2$ is a prime ideal in $\mathfrak{A}_2$.

Since $1 \in X$, $1 \notin J_2$ because $X \cap J_2 = \emptyset$. Suppose $x \in J_2$ and $y \leq_2 x$, then obviously $y \in \downarrow_2(J_2) = J_2$. Suppose $x, y \in J_2 = \downarrow_2(J_2)$. Then, since $x \cup y \leq_2 x \cup y$, $x \cup y \in J_2$. Thus $J_2$ is a proper ideal. In order to be shown that $J_2$ is prime, suppose $x \cap y \in J_2$, $x \notin J_2$, and $y \notin J_2$. Because $J_2$ is a maximal element in $\mathfrak{J}_2$, there exists an element $p \in X \cap \downarrow_2(J_2 \cup \{x\})$, which is not empty. Then, since $J_2 = \downarrow_2(J_2)$, there is $u \in J_2$ such that $a \leq_1 p \leq_2 u \cup x$. Similarly, there exists an element $q \in X \cap \downarrow_2(J_2 \cup \{y\})$, which is not empty. Then there is $v \in J_2$ such that $a \leq_1 q \leq_2 v \cup y$. Here $p, q \in A_0$ and $u, v, x, y \in A_2$, and so, $u \cup x, v \cup y \in A_2$. Since $a \leq_1 p \cap q$, $p \cap q \in X$. However, $p \cap q \leq_2 (x \cup u) \cap (y \cup v) = (x \cap y) \cup (x \cap v) \cup (y \cap u) \cup (u \cap y)$, where the last is a join of four elements in $J_2$. Therefore the join is in $J_2$, and so, $p \cap q \in J_2$. This implies that $X \cap J_2 \neq \emptyset$. Hence $J_2 \notin \mathfrak{J}_2$, which is a contradiction. Thus $J_2$ is a prime ideal.

Put $F_2 := A_2 \setminus J_2$. This is, of course a prime filter in $\mathfrak{A}_2$. Put also $F_0 := F_2 \cap A_0$ and $J_0 := J_2 \cap A_0$. Then $X \subseteq F_0$, $Y \subseteq J_0$, and $F_0 \cap J_0 = \emptyset$ hold. Next, consider a family $\mathfrak{F}_1 := \{F \subseteq A_1 \mid F = \uparrow_1 (F),\ \{a\} \cup F_0 \subseteq F,\ F \cap J_0 = \emptyset\}$, where $\uparrow_1(Z) := \{u \in A_1 \mid \bigcap Z_0 \leq_1 u$ for some finite subset $Z_0 \subseteq Z\}$.

Suppose $\uparrow_1 (F_0 \cup \{a\}) \cap J_0 \neq \emptyset$. Then there is $x \in J_0$ such that $x \in \uparrow_1(F_0 \cup \{a\})$. Since $F_0$ is a filter in $\mathfrak{A}_0$, $z \cap a \leq_1 x$ for some element $z \in F_0$. Then $a \leq_1 -z \cup x$, and $z, x \in A_0$, that implies $-z \cup x \in X \subseteq F_0$. Because $z \cap (-z \cup x) \leq_1 x$, $x \in F_0$, which means that $F_0 \cap J_0 \neq \emptyset$. This is a contradiction. Therefore, $\uparrow_1(F_0 \cup \{a\}) \in \mathfrak{F}_1$, and so, $\mathfrak{F}_1 \neq \emptyset$. Take a chain $\mathcal{C} := \{Z_i\}_{i \in \omega}$ of elements in $\mathfrak{F}_1$ and put $Z := \bigcup \mathcal{C}$.

**Claim 3:** $Z$ is a maximal element in $\mathcal{C}$.

For any $u \in \uparrow_1 (Z)$, there are $z_1, \ldots, z_m \in Z$ such that $z_1 \cap \cdots \cap z_m \leq_1 u$. Since $\mathcal{C}$ is a chain, there exists a number $k \in \omega$ such that $z_1, \cdots, z_m \in Z_k$. Then $u \in \uparrow_1 (Z_k) = Z_k \subseteq Z$, which means that $\uparrow_1(Z) = Z$. It is clear that $\{a\} \cup F_0 \subseteq Z$. Suppose there is $v \in Z \cap J_0$. Then $v \in J_0$ and $v \in Z_i$ for some $i \in \omega$, both of which imply that $Z_i \cap J_0 \neq \emptyset$. This is a contradiction. Hence $Z \in \mathfrak{F}_1$. Trivially $Z$ is maximal in $\mathcal{C}$.

By Zorn's lemma again, there exists a maximal element $F_1 \in \mathfrak{F}_1$.

**Claim 4:** $F_1$ is a prime filter in $\mathfrak{A}_1$.

Since $0 \in J_0$ and $F_1 \cap J_0 = \emptyset$, $0 \notin F_1$. Suppose $x \in F_1$ and $x \leq_1 y$. Then, since $x \in F_1$, $y \in \uparrow_1 (F_1) = F_1$. Suppose $x, y \in F_1$, then

$x \cap y \leq_1 x \cap y \in \uparrow_1 (F_1) = F_1$. Hence $F_1$ is a proper filter. For primeness of $F_1$, suppose $x \cup y \in F_1$, $x \notin F_1$ and $y \notin F_1$. Since $F_1$ is maximal in $\mathfrak{F}_1$, there exists an element $p \in \uparrow_1 (F_1 \cup \{x\}) \cap J_0$, that is not empty. Then, since $F_1 = \uparrow_1(F_1)$, there exists $u \in F_1$ such that $u \cap x \leq_1 p$. Similarly there exists an element $q \in \uparrow_1 (F_1 \cup \{y\}$, which is not empty, and there is $v \in F_1$ such that $v \cap y \leq_1 q$. Here, $u, v, x, y \in A_1$ and $(x \cap u) \cup (y \cap y) \leq_1 p \cup q$. Expanding the left hand side, $(x \cup y) \cap (x \cup v) \cap (u \cup y) \cap (u \cup v) \leq_1 p \cup q$. The left hand side is a meet of four elements in $F_1$, which implies that $p \cup q \in F_1$. However, $p, q \in J_0$, and so $p \cup q \in J_0$ must hold. These two contradict to that $F_1 \cap J_0 = \emptyset$. Thus $F_1$ is a prime filter in $\mathfrak{A}_1$.

Now clearly, $a \in F_1$ and $b \notin F_2$. Finally, if $x \in F_1 \cap A_0$, then $x \notin J_0$, and so, $x \notin J_2$, which means that $x \in F_2 \cap A_0$. Conversely, if $x \in F_2 \cap A_0$, then since $F_2 \cap A_0 = F_0$, $x \in F_1 \cap A_0$. Therefore $F_1 \cap A_0 = F_2 \cap A_0$ is proved. □

The equivalence of (SAP) for algebras and (SAPF) for frames is now proved.

**Theorem 4.10**

(1) For a class $\mathcal{K}$ of descriptive frames, if $\mathcal{K}^*$ has the (SAP), then $\mathcal{K}$ has the (SAPF).

(2) For a class $\mathcal{C}$ of modal algebras, if $\mathcal{C}_*$ has the (SAPF), then $\mathcal{C}$ has the (SAP).

**Proof** : (1): Suppose for frames $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$, there are p-morphisms $\mu_1 : \mathcal{F}_1 \to \mathcal{F}_0$ and $\mu_2 : \mathcal{F}_2 \to \mathcal{F}_0$. Then, by Lemma 3.7 $g_1 : \mathcal{F}_0{}^* \to \mathcal{F}_1{}^*$ defined as: $g_1(X) := \{a \in W_1 \mid \mu_1(a) \in X\}$ and $g_2 : \mathcal{F}_0{}^* \to \mathcal{F}_0{}^*$ defined as: $g_2(b) := \{Y \in W_2 \mid \mu_2(b) \in Y\}$ are embeddings. Then, by (AP) of

$\mathcal{K}^*$, there are an algebra $\mathcal{F}^* \in \mathcal{K}^*$ and embeddings $f_1 : \mathcal{F}_1{}^* \to \mathcal{F}^*$ and $f_2 : \mathcal{F}_2{}^* \to \mathcal{F}^*$ such that $f_1 \circ g_1 = f_2 \circ g_2$. As the proof of Theorem 4.8 shows, there exist p-morphisms $\nu_1 : \mathcal{F} \to \mathcal{F}_1$ and $\nu_2 : \mathcal{F} \to \mathcal{F}_2$ such that $\mu_1 \circ \nu_1 = \mu_2 \circ \nu_2$, where $\nu_1(x) := \tau_1{}^{-1}(\{X \in P_1 \mid f_1(X) \in \tau_1(x)\})$ and $\nu_2(y) := \tau_2{}^{-1}(\{Y \in P_2 \mid f_2(Y) \in \tau_2(y)\})$, for $x, y \in W$.

To be proved the contraposition of (SAPF) of $\mathcal{K}$, suppose for $x \in W_1, y \in W_2$, $\nu_1{}^{-1}(x) \cap \nu_2{}^{-1}(y) = \emptyset$ in $\mathcal{F}$. Put $\mathcal{S} := \{f_1(X) \cap f_2(Y) \mid X \in P_1, Y \in P_2, X \in \tau_1(x), Y \in \tau_2(y)\}$

**Claim:** $\bigcap \mathcal{S} = \emptyset$.

Because, suppose otherwise. Then there exists a point $a \in \bigcap\{f_1(X) \cap f_2(Y) \mid X \in P_1, Y \in P_2, x \in X, y \in Y\}$. This means that for any $X \in P_1$ and for any $Y \in P_2$, if $x \in X$ and $y \in Y$, then $a \in f_1(X) \cap f_2(Y)$. This can be rewritten into:

$\forall X \in P_1$ and $\forall Y \in P_2\big[\big(x \in X$ implies $\nu_1(a) \in X\big)$ and $\big(y \in Y$ implies $\nu_2(a) \in Y\big)\big]$. $\cdots\cdots\cdots(\natural)$

However, by the assumption $\nu_1{}^{-1}(x) \cap \nu_2{}^{-1}(y) = \emptyset$, either $x \neq \nu_1(a)$ or $y \neq \nu_2(a)$ holds. In the former case, since $\mathcal{F}_1$ is differentiated, there exists $S \in P_1$ such that $x \in S$ and $\nu_1(a) \notin S$. This means that the above condition $(\natural)$ does not hold. Similarly, in the latter case, since $\mathcal{F}_2$ is differentiated, there exists $T \in P_2$ such that $y \in T$ but $\nu_2(a) \notin T$. This means that the above condition $(\natural)$ does not hold. Thus the claim is proved.

Then by compactness of $\mathcal{F}$, the intersection of some finite members in $\mathcal{S}$ is also empty. That is, there are $X_1, X_2, \ldots, X_n \in P_1$ and $Y_1, Y_2, \ldots, Y_n \in P_2$ such that $x \in X_i, y \in Y_i$ for $1 \leq i \leq n$, and $\emptyset = \bigcap_{i=1}^{n} (f_1(X_i) \cap f_2(Y_i)) = f_1(X_1 \cap \cdots \cap X_n) \cap f_2(Y_1 \cap \cdots \cap Y_n)$.

Put $X := X_1 \cap \cdots \cap X_n$ and $Y := Y_1 \cap \cdots \cap Y_n$. Then, $X \in P_1, Y \in P_2, x \in X, y \in Y$, and $f_1(X) \cap f_2(Y) = \emptyset$. The last identity implies

that $f_1(X) \subseteq -f_2(Y) = f_2(-Y)$ in $\mathcal{F}^*$. By (SAP) for $\mathcal{K}^*$, there exists $Z \in P_0$ such that $X \subseteq g_1(Z)$ and $g_2(Z) \subseteq -Y$, which is $Y \subseteq -g_2(Z)$. Therefore, $x \in X \subseteq g_1(Z)$ and $y \in Y \subseteq -g_2(Z)$ in $\mathcal{F}_0$. Thus, dually, $\mu_1(x) \in Z$ and $\mu_2(y) \notin Z$. Since $\mathcal{F}_0$ is differentiated, $\mu_1(x) \neq \mu_2(y)$.
(2): Suppose there are modal algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$ and embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1, f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. By Lemma 3.6, maps $g_1 : \mathfrak{A}_{1*} \to \mathfrak{A}_{0*}$ defined by $g_1(F) := \{a \in A_0 \mid a \in F\}$ for $F \in F_p(A_1)$ and $g_2 : \mathfrak{A}_{2*} \to \mathfrak{A}_{0*}$ defined by $g_2(G) := \{b \in A_0 \mid b \in G\}$ for $G \in F_p(A_2)$ are p-morphisms. Then by (APF) of $\mathcal{C}_*$, there exist a frame $\mathfrak{A}_*$ and p-morphisms $\sigma_1 : \mathfrak{A}_* \to \mathfrak{A}_{1*}$ and $\sigma_2 : \mathfrak{A}_* \to \mathfrak{A}_{2*}$. Therefore by Theorem 4.8 (2), embeddings $h_1 : \mathfrak{A}_1 \to \mathfrak{A}$ defined by $h_1(a) := \theta^{-1}(\{F \in F_p(A) \mid \sigma_1(F) \in \theta_1(a)\})$, and $h_2 : \mathfrak{A}_2 \to \mathfrak{A}$ defined by $h_2(b) := \theta^{-1}(\{G \in F_p(A) \mid \sigma_2(G) \in \theta_2(b)\})$ satisfy that $h_1 \circ f_1 = h_2 \circ f_2$.

Now, suppose $h_1(a) \leq h_2(b)$ in $\mathfrak{A}$ for $a \in A_1$ and $b \in A_2$. Furthermore, suppose there is no $c \in A_0$ such that both $a \leq_1 f_1(c)$ and $f_2(c) \leq_2 b$ are satisfied. Then, by Lemma 4.9, there exist a prime filter $F_1$ in $\mathfrak{A}_1$ and a prime filter $F_2$ in $\mathfrak{A}_2$ such that $a \in F_1$, $b \notin F_2$ and $F_1 \cap f_1(A_0) = F_2 \cap f_2(A_0)$. The last identity implies that $g_1(F_1) = g_2(F_2)$ in $\mathfrak{A}_0$. Here, by (SAPF) of $\mathcal{C}_*$, there exists a prime filter $H$ in $\mathfrak{A}$ such that $\sigma_1(H) = F_1$ and $\sigma_2(H) = F_2$. Therefore, $a \in \sigma_1(H)$, and so, $H \in \theta(h_1(a))$. However, $b \notin \sigma_2(H)$, which means that $H \notin \theta(h_2(b))$. Thus $\theta(h_1(a)) \not\subseteq \theta(h_2(b))$, which contradicts to the assumption. $\square$

# 5 An algebraic method to prove the CIP for modal logics

There are several syntactical methods of proving the CIP for modal logics. Among them, Maehara method based on cut eliminable sequent calculus ([19]) and inseparable tableaux method (for example,[4]) using semantic tableaux are two major ones. In this section, the latter method is rewritten into an algebraically equivalent style in order to apply the method to a wider class of modal logics.

**Theorem 5.1** Let $\mathcal{V}$ be a variety of modal algebras and $\mathbf{L} := \mathbf{L}(\mathcal{V})$ the modal logic corresponding to $\mathcal{V}$.

(1) For modal algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$, suppose that there exist embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. Then there exist a *modal algebra* $\mathfrak{A}^\sharp$ and embeddings $g_1 : \mathfrak{A}_1 \to \mathfrak{A}^\sharp$ and $g_2 : \mathfrak{A}_2 \to \mathfrak{A}^\sharp$ which satisfy the following:

   (a) $(g_1 \circ f_1)(x) = (g_2 \circ f_2)(x)$ for any $x \in A_0$.

   (b) For any $x \in A_1$ and for any $y \in A_2$, if $g_1(x) \leq g_2(y)$ in $\mathfrak{A}$, then there exists an element $z \in A_0$ such that $x \leq_1 f_1(z)$ in $\mathfrak{A}_1$ and $f_2(z) \leq_2 y$ in $\mathfrak{A}_2$.

(2) In (1), if $\mathfrak{A}^\sharp \in \mathcal{V}$, then $\mathbf{L}$ has the CIP.

**Proof** : (1): For modal algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$, suppose that there exist embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. Here both $f_1$ and $f_2$ may be assumed to be identity maps, that is, $f_1(x) = f_2(x) = x$ holds for $x \in A_0$. Define an algebra $\mathfrak{A}^\sharp := \langle \mathcal{P}(W), \cap, \cup, -, I_R, \emptyset, W \rangle$, where: $W := \{\langle F, J \rangle \mid F$ is a prime filter in $\mathfrak{A}_1, J$ is a prime ideal in$\mathfrak{A}_2$, $(F \cap A_0) \cap (J \cap A_0) = \emptyset, (F \cap A_0) \cup (J \cap A_0) = W_0\}$, and ${}_{\langle F_1, J_1 \rangle} R_{\langle F_2, J_2 \rangle}$, if for any $I_1(a) \in A_1$, $I_1(a) \in F_1$ implies $a \in F_2$ and for any $I_2(b) \in A_2$, $I_2(b) \notin J_1$ implies $b \notin J_2$. From this relation $R$, the operation $I_R$ is defined as in Proposition 3.5. It is easily seen that this $\mathfrak{A}^\sharp$ is indeed a modal algebra. Furthermore, define maps $g_1 : \mathfrak{A}_1 \to \mathfrak{A}^\sharp$ and $g_2 : \mathfrak{A}_2 \to \mathfrak{A}^\sharp$ as: $g_1(x) := \{\langle F, J\rangle \in W \mid x \in F\}$ for $x \in A_1$, $g_2(y) := \{\langle F, J\rangle \in W \mid y \notin J\}$ for $y \in A_2$.

**Claim 1:** $g_1$ is one to one.

Suppose $x \not\leq_1 y$ in $\mathfrak{A}_1$. Then, there exists a prime filter $F_1$ in $\mathfrak{A}_1$ such that $x \in F_1$ and $y \notin F_1$. Put $J_1 := A_1 \setminus F_1, F_0 := F_1 \cap A_0$ and $J_0 := J_1 \cap A_0$. Consider a family $\mathfrak{J} := \{J \subseteq A_2 \mid J = \downarrow_2 (J), J_0 \subseteq J, J \cap F_0 = \emptyset\}$. It is obvious that $\downarrow_2 (\downarrow_2 (J_0)) = \downarrow_2 (J_0)$ and that $J_0 \subseteq \downarrow_2 (J_0)$. Suppose that $\downarrow_2 (J_0) \cap F_0 \neq \emptyset$. Then, there is $a \in F_0$ and there are $b_1, \ldots, b_n \in J_0$ such that $a \leq_2 b_1 \cap \cdots \cap b_n$. Since $F_0 \subseteq F_1$, that is a prime filter, and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$, $b_1 \cap \cdots \cap b_n \in F_1 \cap A_0 = F_0$, and so, $b_i \in F_0$ for some $i$. Therefore $J_0 \cap F_0 \neq \emptyset$, which is a contradiction. Hence $\downarrow_2 (J_0) \cap F_0 = \emptyset$. Thus $\mathfrak{J} \neq \emptyset$.

Take a chain $\mathcal{C} := \{Z_i\}_{i \in \omega}$ of elements in $\mathfrak{J}$, and put $Z := \bigcap \mathcal{C}$. Then for $u \in \downarrow_2 (Z)$, there exist $v_1, \ldots, v_m \in Z$ such that $u \leq_2 v_1 \cup \cdots \cup v_m$.Because $\mathcal{C}$ is a chain, $v_1, \ldots, v_m \in Z_j$ for some $j$. Therefore $u \in \downarrow_2 (Z_j) = Z_j \subseteq Z$, which implies that $\downarrow_2 (Z) = Z$. It is easily seen that $J_0 \subseteq Z$ and $Z \cap F_0 = \emptyset$. Thus $Z \in \mathfrak{J}$. Now, by Zorn's lemma $\mathfrak{J}$ has a maximal element $J_2$.

This $J_2$ is a prime ideal in $\mathfrak{A}_2$. Because, by the facts $1 \in F_0$ and $J_2 \cap F_0 = \emptyset$, $1 \notin J_2$ follows. Suppose $a \in J_2$ and $b \leq_2 a$, then $b \in \downarrow_2 (J_2) = J_2$. Suppose $a, b \in J_2$, since $a \cup b \leq_2 a \cup b$, $a \cup b \in \downarrow_2 (J_2) = J_2$. Hence $J_2$ is a proper ideal. To be seen that it is prime, suppose $a \cap b \in J_2$, and $a \notin J_2$ and $b \notin J_2$. Since $J_2$ is maximal in $\mathfrak{J}$, there exists $p$ in $\downarrow_2 (J_2 \cup \{a\}) \cap F_0 \neq \emptyset$, and there exists $q$ in

$\downarrow_2 (J_2 \cup \{b\}) \cap F_0 \neq \emptyset$. Since $p, q \in F_0$, $p \cap q \in F_0$. On the other hand, for these $p, q$, there is $u, v \in J_2$ such that $p \leq_2 u \cup a$ and $q \leq_2 v \cup b$ hold. Then, $p \cap q \leq_2 (u \cup a) \cap (v \cup b) = (u \cap v) \cup (u \cap b) \cup (a \cap v) \cup (a \cap b)$. The rightmost hand side is a join of four elements in $J_2$, which is in $J_2$. Therefore $p \cap q \in \downarrow_2 (J_2) = J_2$. This is a contradiction. Hence $J_2$ is a prime ideal.

It is straight forward that $(F_1 \cap A_0) \cap (J_2 \cap A_0) = \emptyset$. Put $F_2 := A_2 \setminus J_2$. Then, if $a \in F_1 \cap A_0 = F_0$, then $a \notin J_2$ and $a \in A_0$, and so $a \in F_2 \cap A_0$. Conversely, if $a \notin F_1 \cap A_0$ but $a \in A_0$, then $a \in J_1 \cap A_0 = J_0 \subseteq J_2$. Therefore $a \notin F_2 \cap A_0$. Thus $F_1 \cap A_0 = F_2 \cap A_0$, and so, $(F_1 \cap A_0) \cup (J_2 \cap A_0) = (F_2 \cap A_0) \cup (J_2 \cap A_0) = (F_2 \cup J_2) \cap A_0 = A_2 \cap A_0 = A_0$. Eventually, $\langle F_1, J_2 \rangle \in W$ is proved. Since $x \in F_1$ but $y \notin F_1$, $\langle F_1, J_2 \rangle \in g_1(x)$ but $\langle F_1, J_2 \rangle \notin g_1(y)$, both of which imply that $g_1(x) \not\subseteq g_2(y)$. This $g_1$ is one to one.

**Claim 2:** $g_1$ is a homomorphism.

It is not so hard to be proved that $g_1(1) = W$, $g_1(-x) = -g_1(x)$, and $g_1(x \cap y) = g_1(x) \cap g_1(y)$. The fact that $g_1(I(x)) = I_R(g_1(x))$ has to be checked. Suppose $\langle F, J \rangle \in g_1(I(x))$. Consider an arbitrary $\langle F', J' \rangle \in W$ such that ${}_{\langle F,J \rangle}R_{\langle F',J' \rangle}$. Then, since $I(x) \in F$, $x \in F'$ which means that $\langle F', J' \rangle \in g_1(x)$ Therefore, $\langle F, J \rangle \in I_R(g_1(x))$.

Conversely suppose $\langle F, J \rangle \notin g_1(I(x))$. Then $I(x) \notin F$. Put $H := \{z \in A_1 \mid I(z) \in F\}$. Then it is obvious to check that $H$ is a proper filter in $\mathfrak{A}_1$ and $x \notin H$. By Lemma 3.3, there exists a prime filter $G_1$ in $\mathfrak{A}_1$ such that $H \subseteq G_1$ and $x \notin G_1$. Put $J_1 := A_1 \setminus G_1$, $G_0 := G_1 \cap A_0$, and $J_0 := J_1 \cap A_0$. Let $K := \{z \in A_2 \mid I(z) \in A_2 \setminus J\}$, and consider a family $\mathfrak{K} := \{G \subseteq A_2 \mid G = \uparrow_2 (G), K \subseteq G, G \cap J_0 = \emptyset\}$. Now, for $a \in \uparrow_2 (K)$, there are $b_1, \ldots, b_n \in K$ such that $b_1 \cap \cdots \cap b_n \leq_2 a$. Since $I(b_1), \ldots, I(b_n) \in A_2 \setminus J$, and $A_2 \setminus J$ is a prime filter in $\mathfrak{A}_2$,

$I(b_1) \cap \cdots \cap I(b_n) = I(b_1 \cap \cdots \cap b_n) \leq_2 I(a) \in A_2 \setminus J$, which implies that $a \in K$, and so $\uparrow_2 (K) = K$. Suppose $K \cap J_0 \neq \emptyset$. Then there is $a \in K \cap J_0$. From the fact that $a \in J_0$, $a \in A_0$ and $a \in J_1 = A_1 \setminus G_1$, which means that $a \notin G_1$. From the fact that $a \in K$, $I(a) \in A_2 \setminus J$, that implies that $I(a) \in F \cap A_0$, and so, $a \in H \subseteq G_1$, but this is a contradiction. Therefore $K \cap J_0 = \emptyset$. Hence $K \in \mathfrak{K} \neq \emptyset$.

Take a chain $\mathcal{C} := \{Z_i\}_{i \in \omega}$ of elements in $\mathfrak{K}$, and put $Z := \bigcap \mathcal{C}$. Then for $a \in \uparrow_2 (Z)$, there exist $b_1, \ldots, b_m \in Z$ such that $b_1 \cap \cdots \cap b_m \leq_2 a$.Because $\mathcal{C}$ is a chain, $b_1, \ldots, b_m \in Z_j$ for some $j$. Therefore $a \in \uparrow_2 (Z_j) = Z_j \subseteq Z$, which implies that $\uparrow_2 (Z) = Z$. It is easily seen that $K \subseteq Z$ and $Z \cap J_0 = \emptyset$. Thus $Z \in \mathfrak{K}$. Now, by Zorn's lemma $\mathfrak{K}$ has a maximal element $E_2$.

This $E_2$ is a prime filter in $\mathfrak{A}_2$. Because, by the facts $0 \in J_0$ and $J_0 \cap E_2 = \emptyset$, $0 \notin E_2$ follows. Suppose $a \in E_2$ and $a \leq_2 b$, then $b \in \uparrow_2 (E_2) = E_2$. Suppose $a, b \in E_2$, then since $a \cap b \leq_2 a \cap b$, $a \cap b \in \uparrow_2 (E_2) = E_2$. Thus $E_2$ is a proper filter. To be proved that it is prime, suppose $a \cup b \in E_2$, $a \notin E_2$ and $b \notin E_2$. Then, since $E_2$ is a maximal element in $\mathfrak{K}$, there exists $p$ in $\uparrow_2 (E_2 \cup \{a\}) \cap J_0 \neq \emptyset$ and there exists $q$ in $\uparrow_2 (E_2 \cup \{b\}) \cap J_0 \neq \emptyset$. Since $p, q \in J_0$, $p \cup q \in J_0$. on the other hand, for these $p, q$, there are $u, v \in E_2$ such that $u \cap a \leq_2 p$ and $v \cap b \leq_2 q$ hold. Then, $p \cup q \geq_2 (u \cap a) \cup (v \cap b) = (u \cup v) \cap (u \cup b) \cap (a \cup v) \cap (a \cup b)$. This rightmost hand side is a meet of four elements in $E_2$, which is in $E_2$. Thus $p \cup q \in \uparrow_2 (E_2) = E_2$ This is a contradiction. Therefore $E_2$ is a prime filter. Put $L_2 := A_2 \setminus E_2$. $(E_2 \cap A_0) \cap (J_1 \cap A_0) = E_2 \cap J_0 = \emptyset$, which implies that $(G_1 \cap A_0) \cap (L_2 \cap A_0) = \emptyset$. For any $a \in J_1 \cap A_0 = J_0$, $a \notin E_2$, and so, $a \in L_2 \cap A_0$. For $a \notin J_1 \cap A_0$, since $J_1$ is a prime ideal in $\mathfrak{A}_1$, and $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$, $-a \in J_1 \cap A_0$. Therefore, by the same reasoning, $-a \in L_2 \cap A_0$, which means that

$a \notin L_2 \cap A_0$. Thus $J_1 \cap A_0 = L_2 \cap A_0$. Therefore, $(G_1 \cap A_0) \cup (L_2 \cap A_0) = (G_1 \cap A_0) \cup (J_1 \cap A_0) = (G_1 \cup J_1) \cap A_0 = A_1 \cap A_0 = A_0$. Hence, $\langle G_1, L_2 \rangle \in W$. This pair $\langle G_1, L_2 \rangle$ satisfies the following: For any $I(z) \in A_1$, $I(Z) \in F$ implies $z \in H \subseteq G_1$. For any $I(u) \in A_2$, $I(u) \notin J$ implies $u \in K \subseteq E_2$, and so $u \notin L_2$. These two mean that ${}_{\langle F,J \rangle}R_{\langle G_1, L_2 \rangle}$. However, $x \notin G_1$, that means that $\langle G_1, L_2 \rangle \notin g_1(x)$. Therefore, $\langle F, J \rangle \notin I_R(g_1(X))$. Thus $g_1(I(x)) = I_R(g_1(x))$ is established. It has just been proved that $g_1$ is an embedding. The similar argument can go through for $g_2$ to be seen that $g_2$ also turns out to be an embedding.

Moreover, both $g_1, g_2$ have the properties (a) and (b) as shown below.
(a:) For any $x \in A_0$, $\langle F, J \rangle \in g_1 \circ f_1(x)$ if and only if $f_1(x) = x \in F$ if and only if $x \notin J$ if and only if $\langle F, J \rangle \in g_2(x) = g_2 \circ f_2(x)$. Thus $(g_1 \circ f_1)(x) = (g_2 \circ f_2)(x)$.
(b:) For $x \in A_1$ and $y \in A_2$, suppose there is no $z \in A_0$ such that both $x \leq_1 f_1(z)$ and $f_2(z) \leq_2 y$ hold. Then by Lemma 4.9, there exist a prime filter $F_1$ in $\mathfrak{A}_1$ and a prime filter $F_2$ in $\mathfrak{A}_2$ such that $x \in F_1$ and $y \notin F_2$ and $F_1 \cap A_0 = F_2 \cap A_0$. Put $I_2 := A_2 \setminus F_2$. Obviously $y \in I_2$. It is easily seen that $(F_1 \cap A_0) \cap (I_2 \cap A_0) = \emptyset$, and that $(F_1 \cap A_0) \cup (I_2 \cap A_0) = A_0$. Therefore the pair $\langle F_1, I_2 \rangle \in W$, and $\langle F_1, I_2 \rangle \in g_1(x)$, but $\langle F_1, I_2 \rangle \notin g_2(y)$, all of which imply that $g_1(x) \not\subseteq g_2(y)$.
(2): If $\mathfrak{A}^\sharp \in \mathcal{V}$, then (1) says that the variety $\mathcal{V}$ has the (SAP). Therefore, by Theorem 4.4, $\mathbf{L}(\mathcal{V})$ has the CIP. □

# 6 Outlook

In this note, proofs of the equivalence of frame-theoretic condition and algebraic condition for a normal modal logic to have the Craig's interpolation property or the Halldén completeness are given carefully, based on the duality of modal algebras and general frames. This is a basic in concidering questions about CIP or H-comp for modal logics by way of general frames.

In proving these equivalences, it is realized that descriptiveness of general frames, in particular, differentiatedness and compactness, plays an important role in many aspects. The class of frames which determines a modal logic is not a class of plain frames, but that of frames with such special properties.

It is also realized that this note has quite a few occurrences of the technique to show the existence of a prime filter which is expanded out of a proper filter based on Zorn's lemma. In particular, this technique is used in a proof to show (SAP) of algebras from (SAPF) of frames, and a proof of algebraic method for the CIP in the last section. This suggests that the proof for a modal logic to have the CIP closely resembles the proof of completeness theorem in an algebraic point of view.

The algebraic method presented in the last section possesses enough generality. The question in the next step is, of course, what sort of modal logics satisfy the condition $\mathfrak{A}^\sharp \in \mathcal{V}$. This method is an algebraization of syntactical *Inseparable Tableaux Method*, and it is like a Henkin-style proof of the completeness of modal logics. It will give us a deep understanding of Craig's interpolation property in modal logics to clarify similarities and differences between Kripke complete logics and logics with CIP.

On Halldén completeness, as is seen in the condition (FCH), a frame with a point in it may well be taken to be a *unit* of a semantics. Therefore, it seems that it is not normal modal logics but *quasi normal* modal logics that has to be considered. Then, a question what the normality of a modal logic means in connection with H-comp must be asked after the characterization of quasi normal modal logics with H-comp.

## References

[1] G. Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 31:433–454, 1935.

[2] W. Blok. On the degree of incompleteness in modal logics and the covering relation in the lattice of modal logics. Technical Report 78-07, Department of Mathematics, University of Amsterdam, 1978.

[3] A. Bull. That all normal extensions of S4.3 have the finite model property. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 12:341–343, 1966.

[4] A. Chagrov and M. Zakharyaschev. *Modal Logic.* Oxford University Press, 1997.

[5] K. Fine. The logics containing S4.3. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 17:371–376, 1971.

[6] K. Fine. An incomplete logic containing S4. *Theoria*, 40:23–29, 1974.

[7] K. Fine. Logics containing K4, Part I. *Journal of Symbolic Logic*, 39:229–234, 1974.

[8] K. Fine. Normal Forms in modal logic. *Nortre Dame Journal of Formal Logic*, 16(2):31–42, 1975.

[9] K. Fine. Logics containing K4, Part II. *Journal of Symbolic Logic*, 50:619–651, 1985.

[10] D. M. Gabbay. Semantic proof of the Craig's interpolation theorem for intuitionistic logics and extensions, I, II. In R.O.Gandy and C.M.E. Yates, editors, *Logic Colloquium '69*, pages 391–401, 403–410. North Holland, Amsterdam, 1971.

[11] D. M. Gabbay. Craig's interpolation theorem for modal logics. In W. Hodges, editor, *Proceedings of logic conference, London 1970*, volume 255 of *Lecture Notes in Mathematics*, pages 111–127. Springer Verlag, Berlin, 1972.

[12] I. L. Humberstone and J. van Benthem. Halldén-completeness by gluing of kripke frames. *Notre Dame Journal of Formal Logic*, 24:426–430, 1983.

[13] B. Jónsson and A. Tarski. Boolean Algebras with Operators. Part I. *American Journal of Mathematics*, 73(4):891–939, 1951.

[14] B. Jónsson and A. Tarski. Boolean Algebras with Operators. Part II. *American Journal of Mathematics*, 74(1):127–162, 1952.

[15] E. J. Lemmon. A note on Halldén incompleteness. *Notre Dame Journal of Formal Logic*, 7:296–300, 1966.

[16] L. L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras. *Algebra and Logic*, 16:427–455, 1977.

[17] L. L. Maksimova. Interpolation theorems in modal logic and amalgamable varieties of topological Boolean algebras. *Algebra and Logic*, 18:348–370, 1979.

[18] Y. Miyazaki. Kripke incomplete logics containing KTB. *Studia Logica*, 85:311–326, 2007.

[19] H. Ono. Proof-theoretic methods in non-classical logic – an introduction. In *Theories of Types and Proofs*, (MSL Memoirs 2), pages 207–254. Mathematical Society of Japan, Tokyo, 1998.

[20] G. F. Schumm. Some failures of interpolation in modal logic. *Notre Dame Journal of Formal Logic*, 27:108–110, 1986.

[21] S. K. Thomason. An incomplete theorem in modal logic. *Theoria*, 40:30–34, 1974.

[22] J. van Benthem. Two simple incomplete modal logics. *Theoria*, 44:25–37, 1978.